2C9
Design for seismic and climate changes

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List of lectures

1. Elements of seismology and seismicity I
2. Elements of seismology and seismicity II
3. Dynamic analysis of single-degree-of-freedom systems I
4. Dynamic analysis of single-degree-of-freedom systems II
5. Dynamic analysis of single-degree-of-freedom systems III
6. Dynamic analysis of multi-degree-of-freedom systems
7. Finite element method in structural dynamics I
8. Finite element method in structural dynamics II
9. Earthquake analysis I

Finite element method in structural dynamics II
Numerical evaluation of dynamic response

1. Dynamic response analysis
2. Time-stepping procedure
3. Central difference method
4. Newmark's method
5. Stability and computational error of time integration schemes
6. Example – direct integration – central difference method
7. Analysis of nonlinear response – average acceleration method
8. HHT method

1. DYNAMIC RESPONSE ANALYSIS

- The equation of motion represents *N* differential equations

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)$$

N equations

- Considering modal analysis, the previous equations can be reduced if the the nodal displacements are approximated by a linear combination of the first J natural modes (usually J much less then N, number of DOF)

$$\mathbf{u}(t) \approx \sum_{n=1}^{J} \boldsymbol{\phi}_n q_n(t) = \boldsymbol{\Phi}\mathbf{q}(t)$$

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \ldots & \boldsymbol{\phi}_J \end{bmatrix} \qquad \mathbf{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ \ldots \\ q_J(t) \end{Bmatrix}$$

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}(t)$$

J equations *(J<<N)*

$$\mathbf{M} = \boldsymbol{\Phi}^T\mathbf{m}\boldsymbol{\Phi}\ ;\ \mathbf{C} = \boldsymbol{\Phi}^T\mathbf{c}\boldsymbol{\Phi}\ ;\ \mathbf{K} = \boldsymbol{\Phi}^T\mathbf{k}\boldsymbol{\Phi}\ ;\ \mathbf{P}(t) = \boldsymbol{\Phi}^T\mathbf{p}(t)$$

(Modal analysis can only be used if the system does not respond into the non-linear range…)

Dynamic response analysis

For systems that behave in a linear elastic fashion…

- If N is small $\rightarrow$ it is appropriate to solve numerically the equations

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)$$

- If N is large $\rightarrow$ it may be advantageous to use modal analysis

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}(t)$$

M and **K** are diagonal matrices

  - For systems with classical damping $\mathbf{C}$ is a diagonal matrix $\rightarrow$ J uncoupled differential equations (see resolution of SDOF systems)

$$M_n\,\ddot{q}_n(t) + C_n\,\dot{q}_n(t) + K_n\,q_n(t) = P_n(t) \qquad n = 1, J$$

  - For systems with nonclassical damping $\mathbf{C}$ is not diagonal and the J equations are coupled $\rightarrow$ use numerical methods to solve the equations

Dynamic response analysis

- The direct solution (using numerical methods) of the *N*x*N* system of equations

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)$$

is adopted in the following situations:

  - For systems with few degrees of freedom
  - For systems and excitations where most of the modes contribute significantly to the response
  - For systems that respond into the non-linear range

- Numerical methods can also be used within the modal analysis for system with nonclassical damping

2. TIME-STEPPING PROCEDURE

- Equations of motion for linear MDOF system excited by force vector $\mathbf{p}(t)$ or earthquake-induced ground motion $\ddot{u}_g(t)$

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t) \quad \text{or} \quad -\mathbf{m}\iota\ddot{u}_g(t)$$

initial conditions $\mathbf{u}(0) = \mathbf{u}_0$ and $\dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0$

time scale is divided into a series of time steps, usually of constant duration

$$\Delta t = t_{i+1} - t_i$$

$$\mathbf{p}_i \equiv \mathbf{p}(t_i) \quad \mathbf{u}_i \equiv \mathbf{u}(t_i) \quad \dot{\mathbf{u}}_i \equiv \dot{\mathbf{u}}(t_i) \quad \ddot{\mathbf{u}}_i \equiv \ddot{\mathbf{u}}(t_i)$$

$$\boxed{\mathbf{m}\ddot{\mathbf{u}}_i + \mathbf{c}\dot{\mathbf{u}}_i + \mathbf{k}\mathbf{u}_i = \mathbf{p}_i \quad \Rightarrow \quad \mathbf{m}\ddot{\mathbf{u}}_{i+1} + \mathbf{c}\dot{\mathbf{u}}_{i+1} + \mathbf{k}\mathbf{u}_{i+1} = \mathbf{p}_{i+1}}$$

unknown vectors $\mathbf{u}_{i+1} \quad \dot{\mathbf{u}}_{i+1} \quad \ddot{\mathbf{u}}_{i+1}$

Explicit methods - equations of motion are used at time instant *i*
Implicit methods - equations of motion are used at time instant *i*+1

Time-stepping procedure

- Modal equations for linear MDOF system excited by force vector $\mathbf{p}(t)$

$$\mathbf{u}(t) \cong \sum_{n=1}^{J} \phi_n q_n(t) = \mathbf{\Phi}\mathbf{q}(t)$$

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{P}(t)$$

$$\mathbf{M} = \mathbf{\Phi}^T\mathbf{m}\mathbf{\Phi} \quad \mathbf{C} = \mathbf{\Phi}^T\mathbf{c}\mathbf{\Phi} \quad \mathbf{K} = \mathbf{\Phi}^T\mathbf{k}\mathbf{\Phi} \quad \mathbf{P}(t) = \mathbf{\Phi}^T\mathbf{p}(t)$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}}_i + \mathbf{C}\dot{\mathbf{q}}_i + \mathbf{K}\mathbf{q}_i = \mathbf{P}_i \;\Rightarrow\; \mathbf{M}\ddot{\mathbf{q}}_{i+1} + \mathbf{C}\dot{\mathbf{q}}_{i+1} + \mathbf{K}\mathbf{q}_{i+1} = \mathbf{P}_{i+1}}$$

<u>unknown vectors</u> $\quad \mathbf{q}_{i+1} \quad \dot{\mathbf{q}}_{i+1} \quad \ddot{\mathbf{q}}_{i+1}$

3. CENTRAL DIFFERENCE METHOD

explicit integration method

dynamic equilibrium condition

for direct solution $x_n \rightarrow \mathbf{u_i}$ $f_n \rightarrow \mathbf{p_i}$ $M,\ C,\ K \rightarrow \mathbf{m,\ c,\ k}$

for modal analysis $x_n \rightarrow \mathbf{q_i}$ $f_n \rightarrow \mathbf{P_i}$ $M,\ C,\ K \rightarrow \mathbf{M,\ C,\ K}$

$$\dot{x}_n = \frac{x_{n+1} - x_{n-1}}{2\Delta t}$$

$$\ddot{x}_n = \frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2}$$

$\Rightarrow$

$$M\ddot{x}_n + C\dot{x}_n + Kx_n = f_n$$

$$M\left[\frac{x_{n+1} - 2x_n + x_{n-1}}{\Delta t^2}\right] + C\left[\frac{x_{n+1} - x_{n-1}}{2\Delta t}\right] + Kx_n = f_n$$

$\Downarrow$

$$x_{n+1} = \underbrace{\left[\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right]}_{\hat{K}}^{-1}\left[f_n - \underbrace{\left(K - \frac{2}{\Delta t^2}M\right)}_{b}x_n - \underbrace{\left(\frac{1}{\Delta t^2}M - \frac{1}{2\Delta t}C\right)}_{a}x_{n-1}\right]$$

Central difference method

for diagonal M and C, finding an inverse of $\left[\frac{1}{\Delta t^2}M + \frac{1}{2\Delta t}C\right]^{-1}$ is trivial (uncoupled equations)

time step

conditionally stable method - stability is controlled by the highest frequency or shortest period T_M (function of the element size used in the FEM model)

$$\Delta t \leq \frac{T_M}{\pi}$$

the time step should be small enough to resolve the motion of the structure. For modal analysis it is controlled by the highest mode with the period T_J

$$\Delta t \approx \frac{T_J}{20}$$

the time step should be small enough to follow the loading function – acceleration records are typically given at constant time increments (e.g. every 0.02 seconds) which may also influence the time step.

Central difference method

CENTRAL DIFFERENCE METHOD: LINEAR SYSTEMS

1.0 *Initial calculations*

1.1 $(q_n)_0 = \dfrac{\phi_n^T \mathbf{m} \mathbf{u}_0}{\phi_n^T \mathbf{m} \phi_n}$; $(\dot{q}_n)_0 = \dfrac{\phi_n^T \mathbf{m} \dot{\mathbf{u}}_0}{\phi_n^T \mathbf{m} \phi_n}$. (initial conditions)

$$\mathbf{q}_0^T = \langle (q_1)_0, \ldots, (q_J)_0 \rangle \qquad \dot{\mathbf{q}}_0^T = \langle (\dot{q}_1)_0, \ldots, (\dot{q}_J)_0 \rangle$$

1.2 $\mathbf{P}_0 = \mathbf{\Phi}^T \mathbf{p}_0$.

1.3 Solve: $\mathbf{M}\ddot{\mathbf{q}}_0 = \mathbf{P}_0 - \mathbf{C}\dot{\mathbf{q}}_0 - \mathbf{K}\mathbf{q}_0 \Rightarrow \ddot{\mathbf{q}}_0$.

1.4 Select Δt.

1.5 $\mathbf{q}_{-1} = \mathbf{q}_0 - \Delta t\, \dot{\mathbf{q}}_0 + \dfrac{(\Delta t)^2}{2} \ddot{\mathbf{q}}_0$.

1.6 $\hat{\mathbf{K}} = \dfrac{1}{(\Delta t)^2}\mathbf{M} + \dfrac{1}{2\Delta t}\mathbf{C}$. (effective stiffness matrix)

1.7 $\mathbf{a} = \dfrac{1}{(\Delta t)^2}\mathbf{M} - \dfrac{1}{2\Delta t}\mathbf{C}$; $\mathbf{b} = \mathbf{K} - \dfrac{2}{(\Delta t)^2}\mathbf{M}$.

Central difference method

2.0 *Calculations for each time step i*

2.1 $\mathbf{P}_i = \mathbf{\Phi}^T \mathbf{p}_i$.

2.2 $\hat{\mathbf{P}}_i = \mathbf{P}_i - \mathbf{a}\mathbf{q}_{i-1} - \mathbf{b}\mathbf{q}_i$. (effective load vector)

2.3 Solve: $\hat{\mathbf{K}}\mathbf{q}_{i+1} = \hat{\mathbf{P}}_i \Rightarrow \mathbf{q}_{i+1}$.

2.4 If required:

$$\dot{\mathbf{q}}_i = \frac{1}{2\Delta t}(\mathbf{q}_{i+1} - \mathbf{q}_{i-1}) \qquad \ddot{\mathbf{q}}_i = \frac{1}{(\Delta t)^2}(\mathbf{q}_{i+1} - 2\mathbf{q}_i + \mathbf{q}_{i-1})$$

2.5 $\mathbf{u}_{i+1} = \mathbf{\Phi}\mathbf{q}_{i+1}$.

3.0 *Repetition for the next time step.* Replace i by $i + 1$ and repeat steps 2.1 to 2.5 for the next time step.

for direct solution: delete steps 1.1, 1.2, 2.1 and 2.5
replace (1) $\mathbf{q}$ by $\mathbf{u}$, (2) $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ and $\mathbf{P}$ by $\mathbf{m}$, $\mathbf{c}$, $\mathbf{k}$ and $\mathbf{p}$

4. NEWMARK'S METHOD

implicit integration method

dynamic equilibrium condition

for direct solution $x_n \rightarrow \mathbf{u}_i$ $f_n \rightarrow \mathbf{p}_i$ $M, C, K \rightarrow \mathbf{m, c, k}$

for modal analysis $x_n \rightarrow \mathbf{q}_i$ $f_n \rightarrow \mathbf{P}_i$ $M, C, K \rightarrow \mathbf{M, C, K}$

$$x_{n+1} = x_n + \Delta t \dot{x}_n + (0.5-\beta)\Delta t^2 \ddot{x}_n + \beta \ddot{x}_{n+1} \Delta t^2$$
$$\dot{x}_{n+1} = \dot{x}_n + (1-\gamma)\Delta t \ddot{x}_n + \gamma \Delta t \ddot{x}_{n+1}$$

(Newmark's equations)

$$\Rightarrow \quad M\ddot{x}_{n+1} + C\dot{x}_{n+1} + Kx_{n+1} = f_{n+1}$$

("discrete" equations of motion)

$$\Downarrow$$

$$\underbrace{\left[M + \gamma \Delta t\, C + \beta \Delta t^2 K\right]}_{\hat{K}} \ddot{x}_{n+1} = \underbrace{f_{n+1} - C\left[\dot{x}_n + (1-\gamma)\Delta t\, \ddot{x}_n\right] - K\left[x_n + \Delta t\, \dot{x}_n + (0.5-\beta)\Delta t^2 \ddot{x}_n\right]}_{\hat{f}_{n+1}}$$

Newmark's method

effective stiffness matrix

$$\hat{K} = [M + \gamma \Delta t C + \beta \Delta t^2 K]$$

effective load vector

$$\hat{f}_{n+1} = f_{n+1} - C\left[\dot{x}_n + (1-\gamma)\Delta t \ddot{x}_n\right] - K\left[x_n + \Delta t \dot{x}_n + (0.5-\beta)\Delta t^2 \ddot{x}_n\right]$$

system of coupled equations

$$\hat{K}\ \ddot{x}_{n+1} = \hat{f}_{n+1} \ \rightarrow\ \ddot{x}_{n+1}$$

unconditionally stable method
for γ = 1/2 and β = 1/4 (average acceleration method)

recommended time step depends on the shortest period of interest $\Delta t \square \frac{T_M}{10}$

Newmark's method

NEWMARK'S METHOD: LINEAR SYSTEMS

1.0 *Initial calculations*

1.1 $(q_n)_0 = \dfrac{\phi_n^T \mathbf{m}\mathbf{u}_0}{\phi_n^T \mathbf{m}\phi_n}$; $(\dot{q}_n)_0 = \dfrac{\phi_n^T \mathbf{m}\dot{\mathbf{u}}_0}{\phi_n^T \mathbf{m}\phi_n}$.

$$\mathbf{q}_0^T = \langle (q_1)_0, \ldots, (q_J)_0 \rangle \qquad \dot{\mathbf{q}}_0^T = \langle (\dot{q}_1)_0, \ldots, (\dot{q}_J)_0 \rangle$$

1.2 $\mathbf{P}_0 = \mathbf{\Phi}^T \mathbf{p}_0$.

1.3 Solve: $\mathbf{M}\ddot{\mathbf{q}}_0 = \mathbf{P}_0 - \mathbf{C}\dot{\mathbf{q}}_0 - \mathbf{K}\mathbf{q}_0 \Rightarrow \ddot{\mathbf{q}}_0$.

1.4 Select Δt.

1.5 $\hat{\mathbf{K}} = \mathbf{K} + \dfrac{\gamma}{\beta \Delta t}\mathbf{C} + \dfrac{1}{\beta(\Delta t)^2}\mathbf{M}$.

1.6 $\mathbf{a} = \dfrac{1}{\beta \Delta t}\mathbf{M} + \dfrac{\gamma}{\beta}\mathbf{C}$; $\mathbf{b} = \dfrac{1}{2\beta}\mathbf{M} + \Delta t\left(\dfrac{\gamma}{2\beta} - 1\right)\mathbf{C}$.

Newmark's method

2.0 *Calculations for each time step i*

2.1 $\mathbf{P}_i = \mathbf{\Phi}^T \mathbf{p}_i$.

2.2 $\Delta\hat{\mathbf{P}}_i = \Delta\mathbf{P}_i + \mathbf{a}\dot{\mathbf{q}}_i + \mathbf{b}\ddot{\mathbf{q}}_i$.

2.3 Solve: $\hat{\mathbf{K}}\Delta\mathbf{q}_i = \Delta\hat{\mathbf{P}}_i \Rightarrow \Delta\mathbf{q}_i$.

2.4 $$\Delta\dot{\mathbf{q}}_i = \frac{\gamma}{\beta\,\Delta t}\Delta\mathbf{q}_i - \frac{\gamma}{\beta}\dot{\mathbf{q}}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mathbf{q}}_i.$$

2.5 $$\Delta\ddot{\mathbf{q}}_i = \frac{1}{\beta(\Delta t)^2}\Delta\mathbf{q}_i - \frac{1}{\beta\Delta t}\dot{\mathbf{q}}_i - \frac{1}{2\beta}\ddot{\mathbf{q}}_i.$$

2.6 $\mathbf{q}_{i+1} = \mathbf{q}_i + \Delta\mathbf{q}_i$, $\dot{\mathbf{q}}_{i+1} = \dot{\mathbf{q}}_i + \Delta\dot{\mathbf{q}}_i$, $\ddot{\mathbf{q}}_{i+1} = \ddot{\mathbf{q}}_i + \Delta\ddot{\mathbf{q}}_i$.

2.7 $\mathbf{u}_{i+1} = \mathbf{\Phi}\mathbf{q}_{i+1}$.

3.0 *Repetition for the next time step*. Replace i by $i + 1$ and implement steps 2.1 to 2.7 for the next time step.

for direct solution: delete steps 1.1, 1.2, 2.1 and 2.7
replace (1) **q** by **u**, (2) **M, C**, **K** and **P** by **m, c, k** and **p**

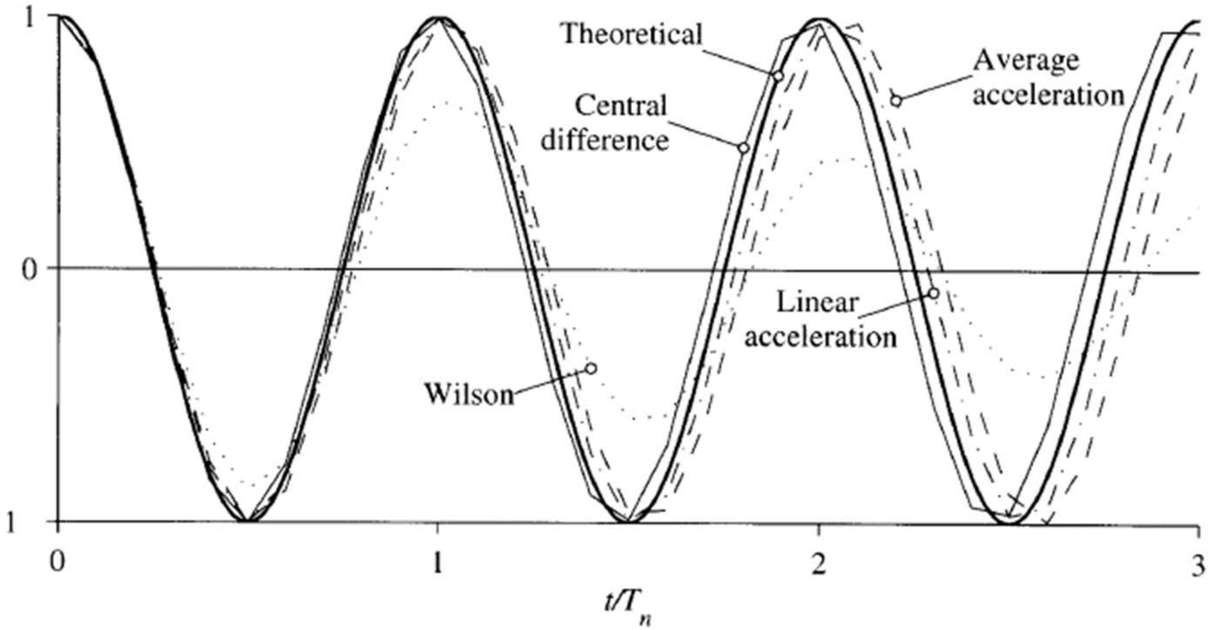
5. STABILITY AND COMPUTATIONAL ERROR OF TIME INTEGRATION SCHEMES

Integration Method	Type of Method	Critical Step Size (Δt_{cr})
Central Different	Explicit	$\frac{2}{\omega}$ $\left(\Delta t \le \frac{T_J}{\pi}\right)$
Newmark Method $\gamma = \frac{1}{2}, \beta = \frac{1}{6}$ (Linear Acceleration)	Implicit	$\frac{3.464}{\omega}$ $(\Delta t \le 0.551 T_J)$
Newmark Method $\gamma = \frac{1}{2}, \beta = \frac{1}{4}$ (Constant-Average-Acceleration)	Implicit	Unconditionally Stable
Newmark Method $\gamma = \frac{1}{2}, \beta = 0$ (Central Difference)	Explicit	$\frac{2}{\omega}$ $\left(\Delta t \le \frac{T_J}{\pi}\right)$
Wilson-θ	Implicit	Unconditionally Stable when $\theta \ge 1.37$

more accurate than

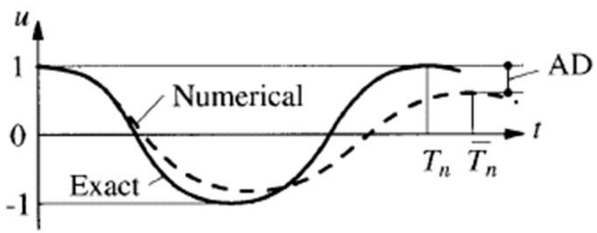
Stability and computational error of time integration schemes

Free vibration problem: $m\ddot{u} + ku = 0 \qquad u(0) = 1 \quad \text{and} \quad \dot{u}(0) = 0 \quad \Rightarrow \quad u(t) = \cos\omega_n t$



$(\Delta t = 0.1T_n)$

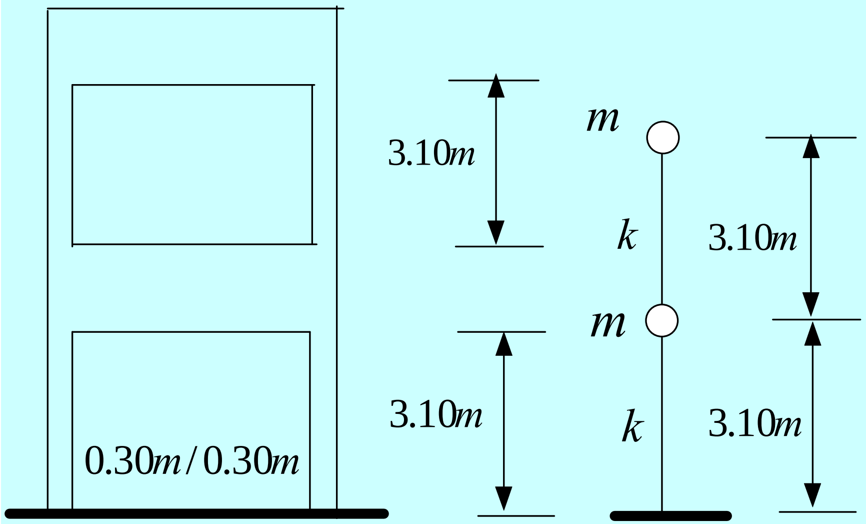
Errors: amplitude decay (AD)
period elongation (PE)

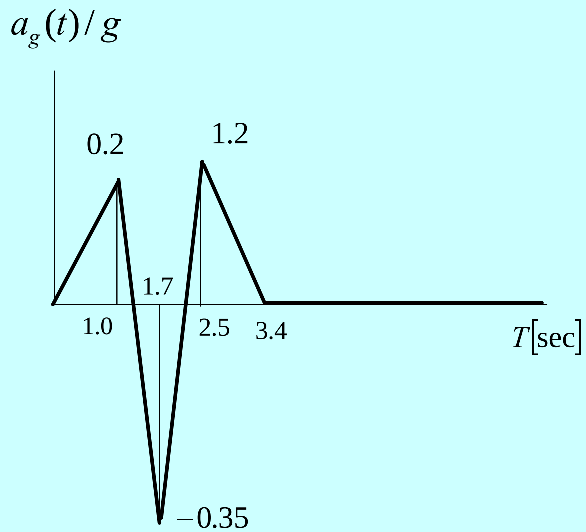


numerical damping?

6. EXAMPLE – DIRECT INTEGRATION – CENTRAL DIFFERENCE METHOD

$$E = 3.43*10^{7}\, kN/m^{2},\ I = \frac{bh^{3}}{12} = \frac{0.3*0.3^{3}}{12} = 6.75*10^{-4}$$





$$k = 2*\frac{12EI}{L^{3}} = 2*\frac{12*3.43*10^{7}*6.75*10^{-4}}{3.1^{4}} = 18640 kN/m$$

$$m = 60 kN\sec^{2}/m$$

$$K = \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix} = \begin{bmatrix} 18640 & -18640 \\ -18640 & 37280 \end{bmatrix} \qquad M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 60 & 0 \\ 0 & 60 \end{bmatrix}$$

$$T_1 = 0.58\sec \qquad T_2 = 0.22\sec$$

Example – direct integration – central difference method

Explicit time stepping scheme

General forcing function

$$x_{n+1} = \left[\frac{1}{\Delta t^2} M\right]^{-1} \left[f_n - \left(K - \frac{2}{\Delta t^2} M\right) x_n - \left(\frac{1}{\Delta t^2} M\right) x_{n-1} \right]$$

$$x_{n+1} = \left[\frac{1}{\Delta t^2} M\right]^{-1} \left(f_n - K x_n\right) + 2 x_n - x_{n-1}$$

(undamped system)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} = \Delta t^2 \begin{bmatrix} \frac{1}{M(1,1)} & 0 \\ 0 & \frac{1}{M(2,2)} \end{bmatrix} \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}_n - \begin{bmatrix} K(1,1) & K(1,2) \\ K(2,1) & K(2,2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n \right\} + 2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n-1}$$

$$\begin{cases} x_{n+1}^1 = \dfrac{\Delta t^2}{M(1,1)} \left\{ f_1 - K(1,1) * x_n^1 - K(1,2) * x_n^2 \right\} + 2 * x_n^1 - x_{n-1}^1 \\ x_{n+1}^2 = \dfrac{\Delta t^2}{M(2,2)} \left\{ f_2 - K(2,1) * x_n^1 - K(2,2) * x_n^2 \right\} + 2 * x_n^2 - x_{n-1}^2 \end{cases}$$

Example – direct integration – central difference method

for the base acceleration case $a(t)$

$$\begin{cases} x_{n+1}^{1} = \dfrac{\Delta t^{2}}{M(1,1)}\left\{-M(1,1)*a_{n} - K(1,1)*x_{n}^{1} - K(1,2)*x_{n}^{2}\right\} + 2*x_{n}^{1} - x_{n-1}^{1} \\ x_{n+1}^{2} = \dfrac{\Delta t^{2}}{M(2,2)}\left\{-M(2,2)*a_{n} - K(2,1)*x_{n}^{1} - K(2,2)*x_{n}^{2}\right\} + 2*x_{n}^{2} - x_{n-1}^{2} \end{cases}$$

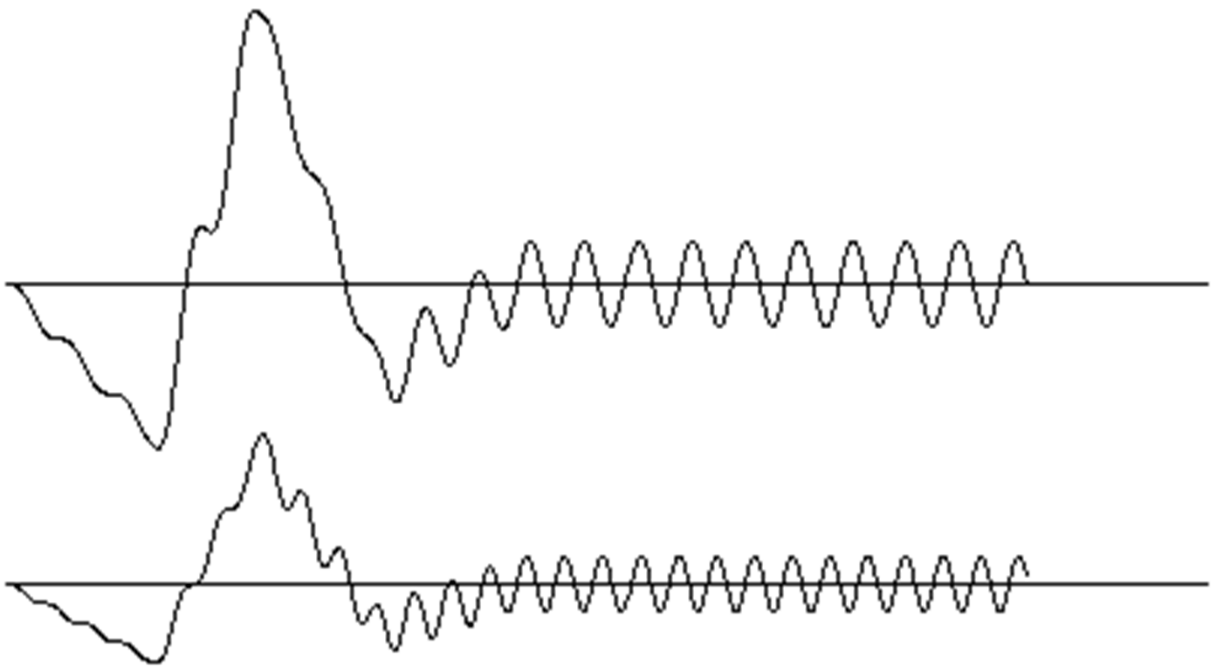
for the problem considered

$$\begin{cases} x_{n+1}^{1} = \dfrac{\Delta t^{2}}{60}\left\{-60*a_{n} - 18640*x_{n}^{1} + 18640*x_{n}^{2}\right\} + 2*x_{n}^{1} - x_{n-1}^{1} \\ x_{n+1}^{2} = \dfrac{\Delta t^{2}}{60}\left\{-60*a_{n} + 18640*x_{n}^{1} - 37280*x_{n}^{2}\right\} + 2*x_{n}^{2} - x_{n-1}^{2} \end{cases}$$

Example – direct integration – central difference method

$\Delta t = 0.01 \sec$

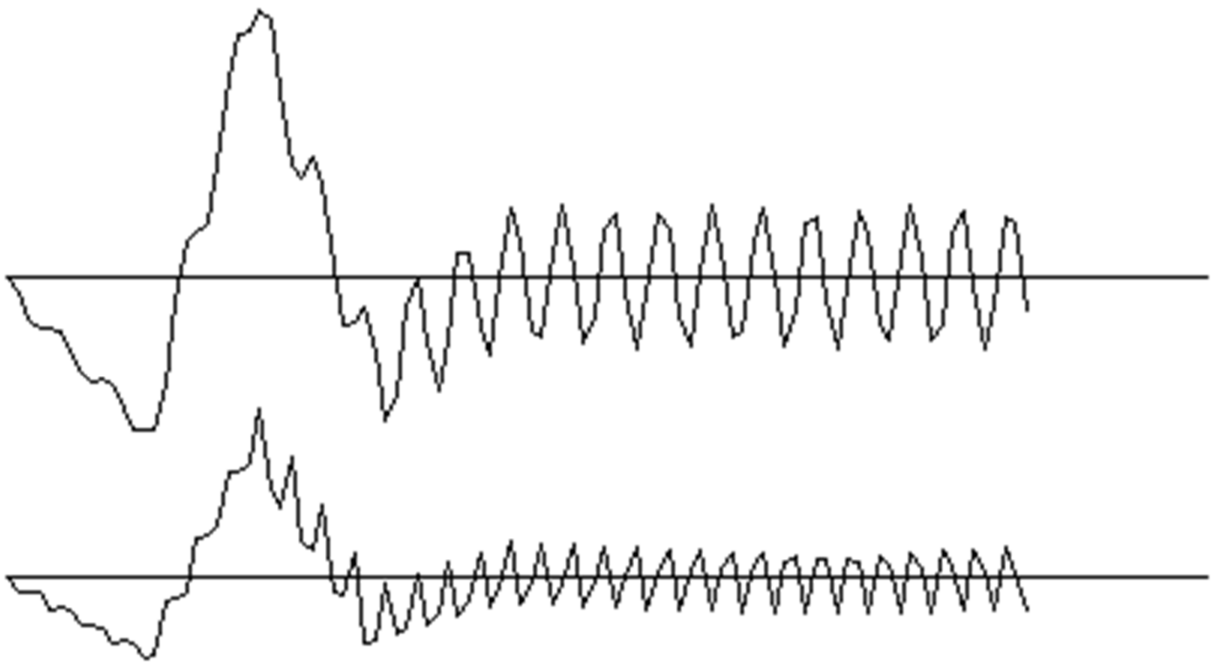
```
U1MAX   1.138665E-02
U2MAX   6.247208E-03
MAX BASE SHEAR   236.0421
Ok
```



Example – direct integration – central difference method

```
U1MAX   1.110161E-02
U2MAX   7.008101E-03
MAX BASE SHEAR   263.9425
Ok
```

$\Delta t = 0.05 \sec$

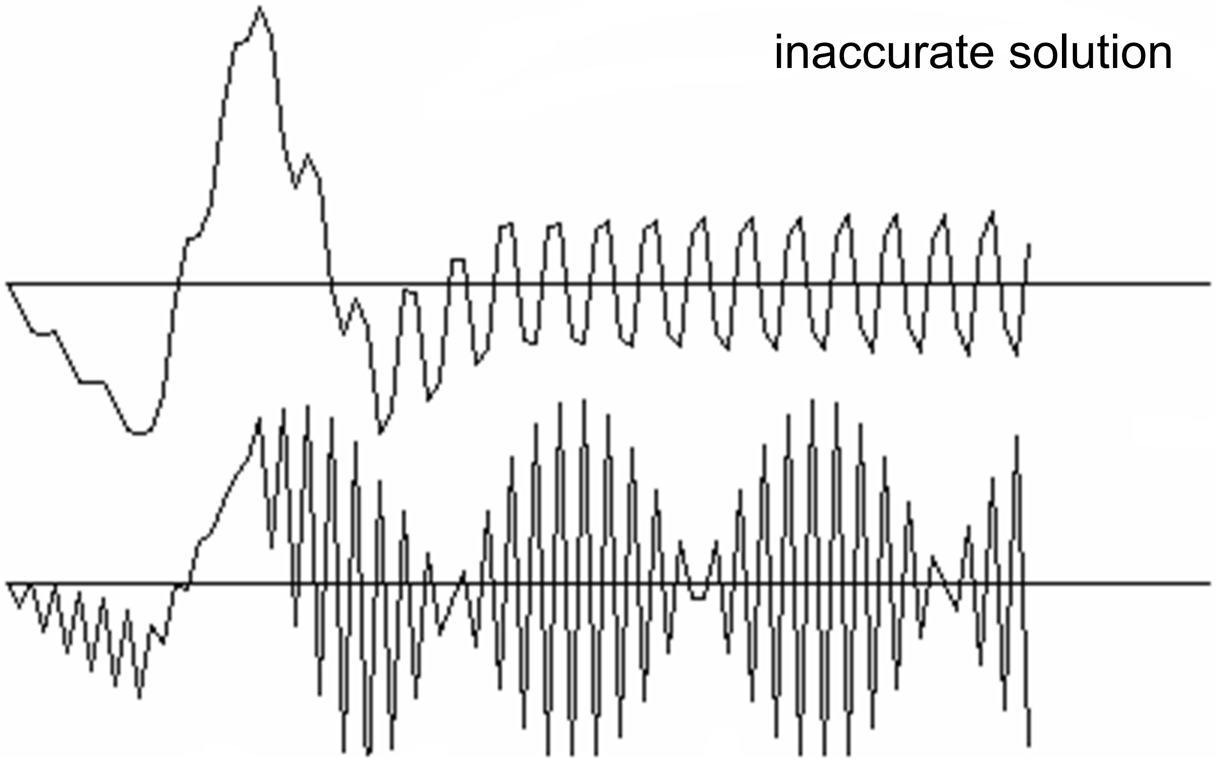


Example – direct integration – central difference method

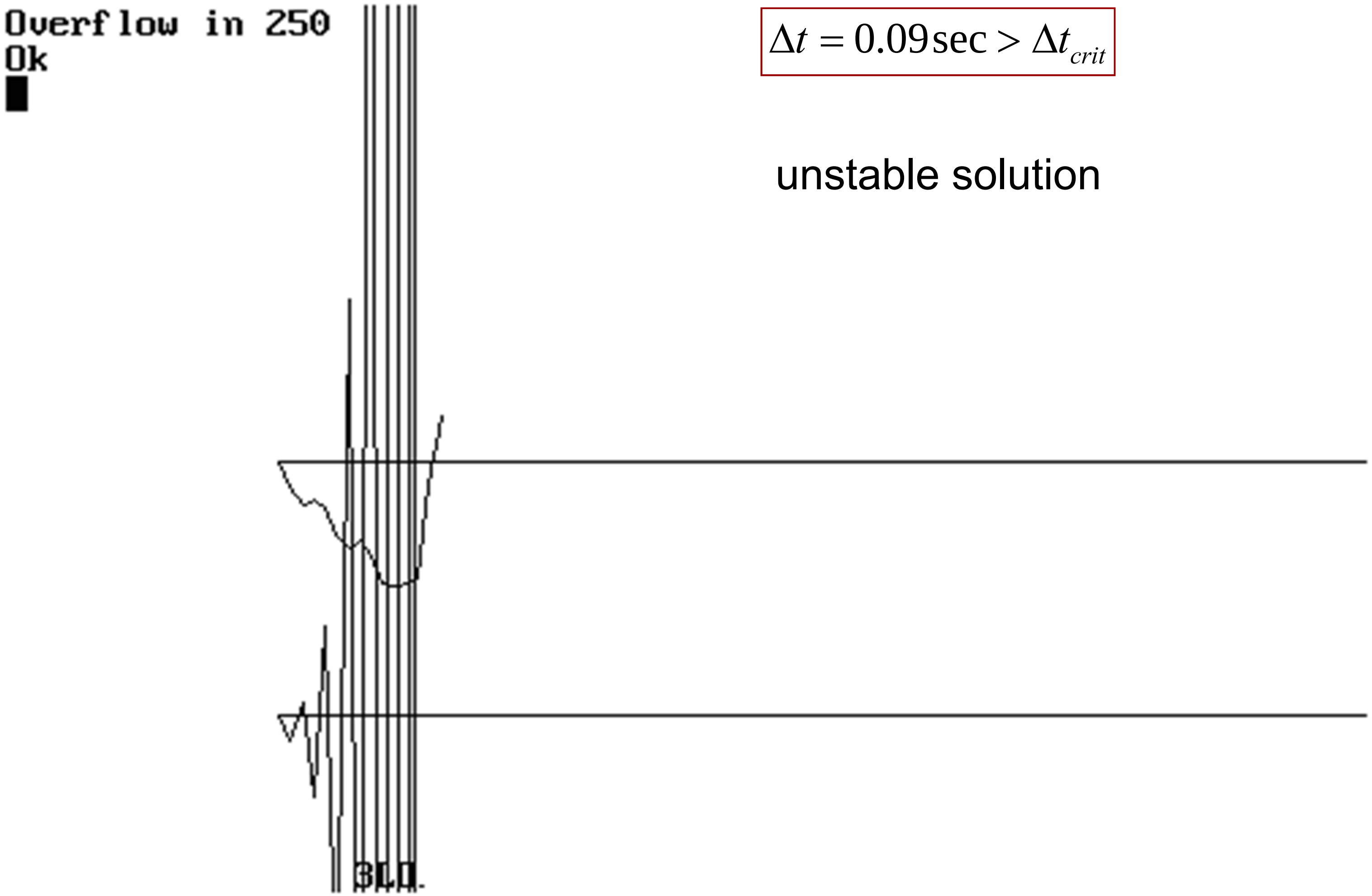
```
U1MAX   1.150803E-02
U2MAX   7.565412E-03
MAX BASE SHEAR   329.8402
Ok
```

$$\Delta t = 0.08\sec > \Delta t_{crit} = \frac{T_{\min}}{\pi} = \frac{0.22}{\pi} \approx 0.07$$

inaccurate solution



Example – direct integration – central difference method



$$\Delta t = 0.09 \sec > \Delta t_{crit}$$

unstable solution

7. ANALYSIS OF NONLINEAR RESPONSE – AVERAGE ACCELERATION METHOD

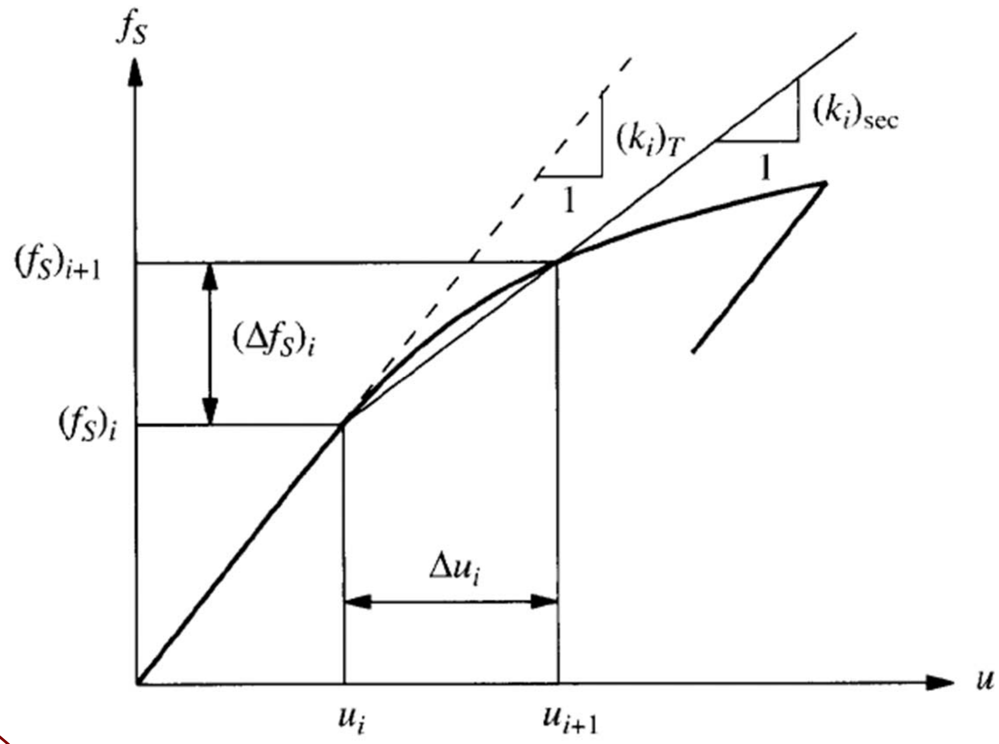
Incremental equilibrium condition

$$\mathbf{m}\Delta\ddot{\mathbf{u}}_i + \mathbf{c}\Delta\dot{\mathbf{u}}_i + \left(\Delta\mathbf{f}_S\right)_i = \Delta\mathbf{p}_i$$

Incremental resisting force

$$\left(\Delta\mathbf{f}_S\right)_i = \left(\mathbf{k}_i\right)_{\text{sec}}\Delta\mathbf{u}_i \approx \underbrace{\left(\mathbf{k}_i\right)_T\Delta\mathbf{u}_i}_{\text{(assumption)}}$$

(assumption)



secant stiffness: $\left(\mathbf{k}_i\right)_{\text{sec}}$
(unknown)

tangent stiffness: $\left(\mathbf{k}_i\right)_T$
(known)

Need for an iterative process within each time step…

Analysis of nonlinear response – average acceleration method

AVERAGE ACCELERATION METHOD: NONLINEAR SYSTEMS

$$\left(\beta = \frac{1}{4} \quad ; \quad \gamma = \frac{1}{2}\right)$$

1.0 *Initial calculations*

1.1 Solve: $\mathbf{m}\ddot{\mathbf{u}}_0 = \mathbf{p}_0 - \mathbf{c}\dot{\mathbf{u}}_0 - (\mathbf{f}_S)_0 \Longrightarrow \ddot{\mathbf{u}}_0$.

1.2 Select Δt.

1.3 $\mathbf{a} = \dfrac{4}{\Delta t}\mathbf{m} + 2\mathbf{c}$; and $\mathbf{b} = 2\mathbf{m}$.

2.0 *Calculations for each time step i*

2.1 $\Delta\hat{\mathbf{p}}_i = \Delta\mathbf{p}_i + \mathbf{a}\dot{\mathbf{u}}_i + \mathbf{b}\ddot{\mathbf{u}}_i$. (effective load vector)

2.2 Determine the tangent stiffness matrix $\mathbf{k}_i$.

2.3 $\hat{\mathbf{k}}_i = \mathbf{k}_i + \dfrac{2}{\Delta t}\mathbf{c} + \dfrac{4}{(\Delta t)^2}\mathbf{m}$. (effective stiffness matrix)

Analysis of nonlinear response – average acceleration method

cont.

2.4 Solve for $\Delta \mathbf{u}_i$ from $\hat{\mathbf{k}}_i$ and $\Delta \hat{\mathbf{p}}_i$ using the iterative procedure of M N-R method

2.5 $\Delta \dot{\mathbf{u}}_i = \frac{2}{\Delta t} \Delta \mathbf{u}_i - 2\dot{\mathbf{u}}_i$.

2.6 $\Delta \ddot{\mathbf{u}}_i = \frac{4}{(\Delta t)^2} \Delta \mathbf{u}_i - \frac{4}{\Delta t} \dot{\mathbf{u}}_i - 2\ddot{\mathbf{u}}_i$.

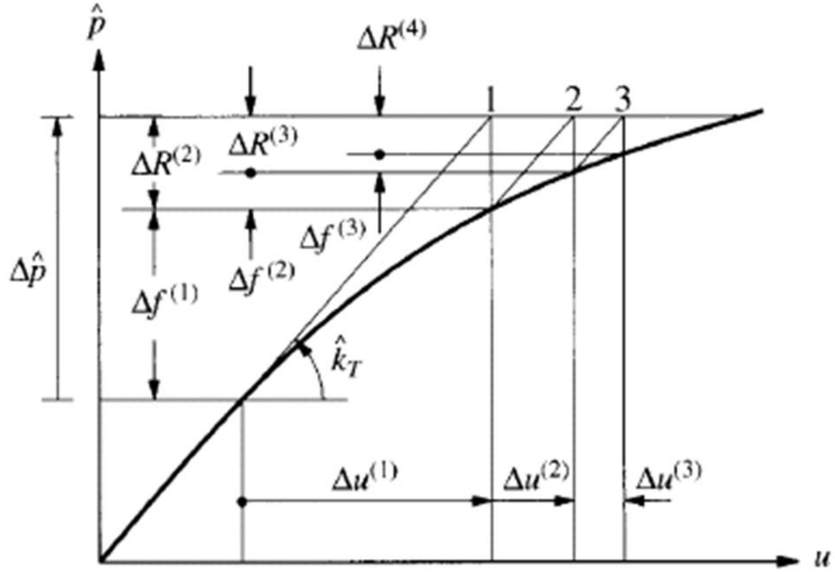
2.7 $\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta \mathbf{u}_i$, $\dot{\mathbf{u}}_{i+1} = \dot{\mathbf{u}}_i + \Delta \dot{\mathbf{u}}_i$, and $\ddot{\mathbf{u}}_{i+1} = \ddot{\mathbf{u}}_i + \Delta \ddot{\mathbf{u}}_i$.

3.0 *Repetition for the next time step.* Replace i by $i + 1$ and implement steps 2.1 to 2.6 for the next time step.

- unconditionally stable method
- no numerical damping
- recommended time step depends on the shortest period of interest

Analysis of nonlinear response – average acceleration method

Modified Newton-Raphson (M N-R) iteration



MODIFIED NEWTON–RAPHSON ITERATION

1.0 *Initialize data.*

$$\mathbf{u}_{i+1}^{(0)} = \mathbf{u}_i \qquad \mathbf{f}_S^{(0)} = (\mathbf{f}_S)_i \qquad \Delta\mathbf{R}^{(1)} = \Delta\hat{\mathbf{p}}_i \qquad \hat{\mathbf{k}}_T = \hat{\mathbf{k}}_i$$

2.0 *Calculations for each iteration,* $j = 1, 2, 3, \ldots$

2.1 Solve: $\hat{\mathbf{k}}_T \, \Delta\mathbf{u}^{(j)} = \Delta\mathbf{R}^{(j)} \Longrightarrow \Delta\mathbf{u}^{(j)}$

2.2 $\mathbf{u}_{i+1}^{(j)} = \mathbf{u}_{i+1}^{(j-1)} + \Delta\mathbf{u}^{(j)}$

2.3 $\Delta\mathbf{f}^{(j)} = \mathbf{f}_S^{(j)} - \mathbf{f}_S^{(j-1)} + (\hat{\mathbf{k}}_T - \mathbf{k}_i)\, \Delta\mathbf{u}^{(j)}$

2.4 $\Delta\mathbf{R}^{(j+1)} = \Delta\mathbf{R}^{(j)} - \Delta\mathbf{f}^{(j)}$

3.0 *Repetition for the next iteration.* Replace j by $j + 1$ and repeat calculation steps 2.1 to 2.4.

To reduce computational effort, the tangent stiffness matrix is not updated for each iteration

8. HILBER-HUGHES-TAYLOR (HHT) METHOD (α – method)

implicit integration method – generalization of Newmark's method
numerical damping of higher frequencies (elimination of high frequency oscillations) + stable and second order convergent

dynamic equilibrium condition

for direct solution $x_n \rightarrow \mathbf{u_i}$ $f_n \rightarrow \mathbf{p_i}$ $M, C, K \rightarrow \mathbf{m, c, k}$

for modal analysis $x_n \rightarrow \mathbf{q_i}$ $f_n \rightarrow \mathbf{P_i}$ $M, C, K \rightarrow \mathbf{M, C, K}$

$$x_{n+1} = x_n + \Delta t \dot{x}_n + (0.5 - \beta)\Delta t^2 \ddot{x}_n + \beta \ddot{x}_{n+1} \Delta t^2$$
$$\dot{x}_{n+1} = \dot{x}_n + (1 - \gamma)\Delta t \ddot{x}_n + \gamma \Delta t \ddot{x}_{n+1}$$

⇒

(Newmark's equations)

⇒

$$M\ddot{x}_{n+1} + (1+\alpha)(C\dot{x}_{n+1} + Kx_{n+1}) - \alpha(C\dot{x}_n + Kx_n) = f(\tilde{t}_{n+1})$$

$\rightarrow \ddot{x}_{n+1}$

("discrete" equations of motion)

(unknowns)

$$\tilde{t}_{n+1} = t_n + (1+\alpha)\Delta t$$

HHT method

3 parameters – values for unconditional stability and second-order accuracy:

$$\beta = (1-\alpha)^2 / 4 \qquad \gamma = \frac{1}{2} - \alpha \qquad -0.3 \leq \alpha \leq 0$$

the smaller value of α – more numerical damping is introduced to the system
for $\alpha = 0$ – Newmark's method with no numerical damping

HHT method for transient analysis of nonlinear problems

$$M\ddot{x}_{n+1} + f_{\text{int}}\left(x_{n+\alpha}, \dot{x}_{n+\alpha}\right) = f\left(t_{n+\alpha}, x_{n+\alpha}\right)$$
$$x_{n+1} = x_n + \Delta t \dot{x}_n + (0.5-\beta)\Delta t^2 \ddot{x}_n + \beta \ddot{x}_{n+1} \Delta t^2$$
$$\dot{x}_{n+1} = \dot{x}_n + (1-\gamma)\Delta t \ddot{x}_n + \gamma \Delta t \ddot{x}_{n+1}$$

system of algebraic equations

$f_{\text{int}}\left(x_{n+\alpha}, \dot{x}_{n+\alpha}\right)$ vector of internal forces
(depends non-linearly on displacements and velocities)

$(\bullet)_{n+\alpha} = (1+\alpha)(\bullet)_{n+1} - \alpha(\bullet)_n$ variables computed by convex combination