COMPUTATIONAL MECHANICS
New Trends and Applications
S.Idelsohn, E. Oñate and E. Dvorkin (Eds.)

NON-LINEAR SMALL STRAIN-SEPARATE EFFECTS SOLUTION OF 3D BEAM SYSTEMS

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Key words: Small strain, geometric non-linearity, 3D beams, axial and shear torque energy, separate deflections effects, iterative solution

Abstract. *The basic energy equations due to axial deformation of 3D bar and beam element with respect to the geometric non-linear deformation based on small strains are briefly derived. The relations between nodal deflections and nodal forces are shown using the geometric stiffness matrix. The improved relations using all terms of the energy expression for the axial deformation are derived. The influence of each component of deflection on the others during the deformation with respect to the non-linear behavior of the element is clearly separated. The basic „geometric non-linear matrices„ are derived. The energy due to deformation caused by St.Venant torsion of 3D-beam element is taken into the account for the geometric non-linear behavior of the 3D element.*

1. INTRODUCTION

Geometric non-linear behavior of structures is of an increasing interest for many researchers. Several research papers have been published in recent years[2, 16, 17, 18.] They are mainly deal with plane frame second order analysis[2, 15.] Work includes plane steel frames with semirigid connections as is in Goto's work [4, 5], or at the works[3, 6.] Toader's work[13] is dealing with semirigid connections of 3D beams. The results of these theoretical work is possible mainly because of computer and software developments. Trends in research on advanced analysis of steel frames are shown at [4, 6, 18]. Space structural frameworks are intensively used since 1980's. Papers dealing with these problems are published in proceedings on Space structures[7, 8], and also the proceedings of Symposium of International Association on Shells and Space Structures[9]. These types of structures allow us to built domes of large spans, where the geometric non-linear behavior is of the significant importance. Stability problems associated with these structures are a field, which is still open for research. The relations for the geometric nonlinear behavior can be expressed in more detail, and recent computer and software developments allows us to solve more complicated problems. Geometric nonlinear behavior of structures described with the finite element method uses a geometric stiffness matrix as is derived by Przemieniecki, at[1]**.** The concept of a geometric stiffness matrix is based on the simplifying assumption that the load applied on the structure is unchanged during the load step increment. The energy expression is based only on the axial deformation of beam fibers. The shear energy due to the torsion of the general 3D beam is not taken into account. Also, the higher order terms in the energy expression due to the deformation in the longitudinal axis are neglected at[1]. The resulting relations, when all these higher order terms are used, are non-linear and depend on cross-sectional properties and material properties of members and nodal displacements. These relations include products of several joint displacements, which lead to an iterative procedure, published at [10, 11]. The subject of the work is to show more general expressions for the geometric non-linear behavior of the 3D beam element and 3D bar element than expressions, which use the geometric stiffness matrix.

The matrix relations, which are usually derived using the higher order nonlinear terms in the energy expression, sometimes looks non-symmetrical[15]. This result is due to the fact that the matrix terms rely on the nodal displacements of the element from the previous load step increment. Basic „higher order stiffness„ matrices, which are based only on the material and cross sectional properties of the member, are derived in the paper. The displacements, which affect the behavior of a structure, are excluded from the matrices. An iterative procedure is recommended for the solution. This approach permits a solution, which is completely separate for high order terms of the energy expression for the 3D beam and bar elements. New cross-sectional properties for the non-linear Saint Venant torsion are derived. The derived solution cans be used with already written finite element computer programs.

2. ASSUMPTIONS

The derivation is based on the following assumptions:

1) 3D members are straight without any imperfections
2) The local coordinate system of the member follows the right hand rule and is coincident

with major principal axis of the member

3) Navier's hypothesis is valid for the cross-section of the member.

4) Torsion is assumed to be Saint Venant type, i. g. warping is neglected.

5) The load step increment is finite and load is constant during the load step

3. The basic relations of non-linear equations

The relation between the nodal displacements and element deformations is described by

$$u\,(x, y, z) = a\ \ U\,(u_1, u_2, u_3, ...u_n) \qquad (1)$$

where u is the vector of the element deformations and U is the vector of nodal displacements. Matrix a is the matrix of functions describing the geometrical relations between these displacements. Non-dimensional coordinates are introduced as $x = \frac{x}{L}, z = \frac{y}{L}, h = \frac{z}{L}$, where x, y, z are dimensions in local coordinate system and L is the length of the 3D element (Fig.1). For a 3D bar we have 6 degree of freedom and for the 3D beam we have 12 degree of freedom. Therefore 3D bar element can resist the axial forces only and the 3D beam element is able to resist axial force, shears in the directions y and z, bending about y and z axis and Saint Venant's torsion about x axis (Fig.2). The curvature of the beam due to bending in one direction is represented by the approximate expression $\frac{1}{r} = \frac{\partial^2 u}{\partial x^2}$, where r is the radius of the curvature of the beam and the u is deflection of the beam in a direction perpendicular to the x axis due to the bending about y or z axis.

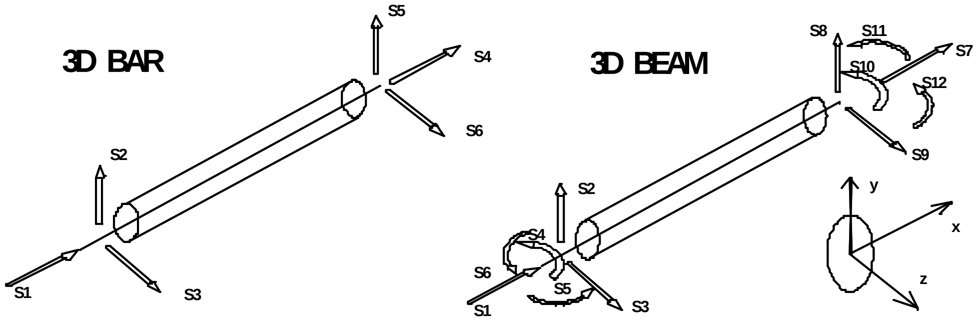

Figure 1

For a 3D bar we have 6 degree of freedom and for the 3D beam we have 12 degrees of freedom. Therefore a 3D bar element can resist the axial forces only and the 3D beam element is able to resist axial force, shears in the y and z directions, bending about y and z axes and Saint Venant’s torsion about x axis (Fig.2). The curvature of the beam due to bending in one direction is represented by the approximate expression $\frac{1}{r} = \frac{\partial^2 u}{\partial x^2}$, where r is the radius of the curvature of the beam and the u is deflection of the beam in a direction perpendicular to the x axis due to the bending about y or z axis. The geometric relations for all these types of loads are derived by Przemieniecki at [1].The matrix a for the bar element and for the beam element are written for the convenience as a^T matrices in the following expression (2).

The strain energy of the element due to its deformation during the finite increment of the external load can be written as in (3).

a$^{\mathbf{T}}$**=** 3D bar element 3D beam element (2)

$$\begin{vmatrix} +(1-x) & 0 & 0 \\ 0 & +(1-x) & 0 \\ 0 & 0 & +(1-x) \\ +x & 0 & 0 \\ 0 & +x & 0 \\ 0 & 0 & +x \end{vmatrix} \qquad \begin{vmatrix} (1-x) & 0 & 0 \\ -6(x^2-x)h & +(2x^3-3x^2+1) & 0 \\ -6(x^2-x)z & 0 & +(2x^3-3x^2+1) \\ 0 & -(1-x)Lz & -(1-x)Lh \\ +(3x^2-4x+1)Lz & 0 & -(x^3-2x^2+x)L \\ -(3x^2-4x+1)Lh & (x^3-2x^2+x)L & 0 \\ x & 0 & 0 \\ +6(x^2-x)h & +(3x^2-2x^3) & 0 \\ +6(x^2-x)z & 0 & +(3x^2-2x^3) \\ 0 & -Lxz & -Lxh \\ +(3x^2-2x)Lz & 0 & -(x^3-x^2)L \\ -(3x^2-2x)Lh & +(x^3-x^2)L & 0 \end{vmatrix}$$

$$U = \tfrac{1}{2} E \int_V e^2 \, dV \qquad (3)$$

here V is the volume of the element, E is Young's modulus and ε is the longitudinal strain. Consider the general 3D beam element as it is subjected to the general nodal loading as shown

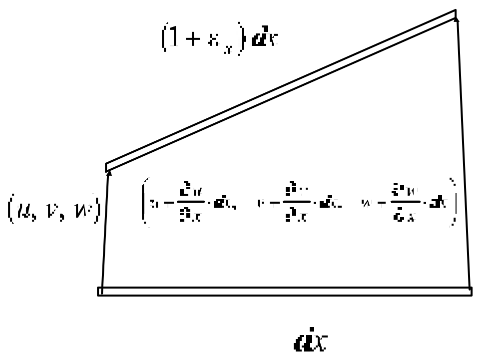

Figure 2

in Fig.1. The deformed length of an infinitesimally small element (Fig.2) can be expressed as $(1+\varepsilon_x^a)$dx where the ε_x^a is the engineering axial strain and dx is the elements length. Applying Pythagora's theorem, the elongation of the element may be expressed as

$$\left[\left(1+\varepsilon_x^a\right)dx\right]^2 = \left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2 + \left(\frac{\partial w}{\partial x}dx\right)^2 \tag{4}$$

This can be simplified as:

$$\varepsilon_x\left(1+\frac{\varepsilon_x}{2}\right) = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{5}$$

The right hand side of equation (5) represents the component of the Green strain tensor e_{xx}.. For the condition of small strain, we can write $(\varepsilon_x^a)^2 = 0$ and the Green tensor coincides with the engineering strain ε_x. We can also neglect the term $\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2$ compared to $\frac{\partial u}{\partial x}$, Therefore, (4) becames

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{6}$$

The curvature of a beam subjected to the bending moment is expressed by

$$\frac{1}{\rho_i} = \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial \varphi_i}{\partial x} \tag{7}$$

where the subscript i refers to curvatures y or z according to the axis about which is the beam bent. The elongation of the fiber of the beam parallel to the longitudinal axis x can be expressed. The curvature strain k_i can be expressed as follows

$$k_x = \frac{\Delta dx}{dx} = \frac{(\rho_z - y)d\varphi_z - dx}{dx} \qquad \text{respectively} \quad k_x = \frac{\Delta dx}{dx} = \frac{(\rho_{y_z} - z)d\varphi_y - dx}{dx} \tag{8}$$

Substituting equation (7) into (8) leads to (9) for the bending strain ε_x^b due to bending at any point y, z from the centroid

$$\varepsilon_x^b = -y\frac{\partial^2 u_y}{\partial x^2} - z\frac{\partial^2 u_z}{\partial x^2} \tag{9}$$

This expression may be added to (6). The results is a final expression for the nonlinear strain ε_x for a 3D beam subjected to axial force, shear and bending about both axis y and z. Since Saint Venant torsion has no effect on the longitudinal strain expression, the final expression for the longitudinal strain is presented as

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - y\frac{\partial^2 u_y}{\partial x^2} - z\frac{\partial^2 u_z}{\partial x^2} \tag{10}$$

Substituting (10) to the strain energy expression (3) results in (11) for potential strain energy. The shape functions from equation (1) are substitute to the expression (11). After that, Castigliano's theorem (part 1) can be applied to the expression (11) with respect to the deflections u_1,, u_2 , ... u_{12} ,. We obtain the relations for the nodal forces S_1 , S_2 , ... S_6 for the 3D bar and S_1 , S_2 ,

.... S_{12} for the 3D bar. The same procedure can be applied to the 3D bar or beam using the proper shape functions from the equation (1). However, expanding this expression is difficult due to its large number of the terms.

$$U=\frac{E}{2}\int_0^L\oint_A\left\{\left(\frac{\partial u_x}{\partial x}\right)^2+\left(\frac{\partial^2 u_y}{\partial x^2}\right)^2 y^2+\left(\frac{\partial^2 u_z}{\partial x^2}\right)^2 z^2+\frac{1}{4}\left(\frac{\partial u_x}{\partial x}\right)^4+\frac{1}{4}\left(\frac{\partial u_z}{\partial x}\right)^4+\frac{\partial u_x}{\partial x}\left(\frac{\partial^2 u_y}{\partial x^2}\right)^2+\frac{\partial u_x}{\partial x}\left(\frac{\partial^2 u_z}{\partial x^2}\right)^2\right. \quad (11)$$

$$-2\frac{\partial u_x}{\partial x}\frac{\partial^2 u_y}{\partial x^2}y-2\frac{\partial u_x}{\partial x}\frac{\partial^2 u_{zy}}{\partial x^2}z+\frac{1}{2}\left(\frac{\partial u_y}{\partial x}\right)^2-\left(\frac{\partial u_y}{\partial x}\right)^2\left(\frac{\partial^2 u_y}{\partial x^2}\right)y-\left(\frac{\partial u_y}{\partial x}\right)^2\left(\frac{\partial^2 u_z}{\partial x^2}\right)z-\frac{\partial^2 u_y}{\partial x^2}\left(\frac{\partial u_z}{\partial x}\right)^2 y$$

$$\left.-\left(\frac{\partial u_z}{\partial x}\right)^2\left(\frac{\partial^2 u_z}{\partial x^2}\right)z+\frac{\partial^2 u_y}{\partial x^2}y\frac{\partial^2 u_z}{\partial x^2}z\right\}dAdx$$

Assume for the 3D bar $\Delta L=(u_4-u_1)$ = constant or for the 3D beam $\Delta L=(u_7-u_1)$ = constant, where is ΔL the elongation of the element during the load step. This assumption is made at [1]. The geometric stiffness matrix $\mathbf{k_G}$ may be derived based on this simplifying assumption. The nodal forces given by the well known equation (12) is the result.

$$\mathbf{S} = (\mathbf{k_E} + \mathbf{k_G})\,\mathbf{U} \quad (12)$$

S is the vector of nodal forces of the element, **U** is the vector of the nodal displacements, $\mathbf{k_E}$ is the elastic stiffness matrix and $\mathbf{k_G}$ is the geometric stiffness matrix.

4. IMPROVED NON-LINEAR SOLUTION FOR THE GEOMETRICAL STIFFNESS

The assumption about the constant length of the element (which implies the constant value of the prestressing force H= EA ΔL in the member) is not made in this paper. The first part of the derivation is presented for the 3D bar element at.[10]. Also basic relation for the 3D beam element, which did not take into the account the terms of the 4th power, was published by Vasek at [11].
The last part of the derivation, which includes terms of the 4th power for the 3D beam element in expression (11) for axial strain energy is derived with the software support. After the described mathematical operation is complete the final expression for the nodal forces of the bar and beam element is determined. The number of the terms which are taken into the account is more then one thousand. The final equation for each force is possible to rewrite in matrix form. The results of the derivation are shown in next paragraphs. The interesting feature of these results is that the expression for the torsional moments S_4 and S_{10} depends on terms which are actually new cross sectional properties of the higher order. These are

$$K_y=\oint_A z^4\,dA, \qquad K_z=\oint_A y^4\,dA, \qquad K_{zy}=\oint_A z^2y^2\,dA, \quad (13)$$

We can call these expressions "moments of inertia of second order". Another feature of this approach is that each nodal force is dependent on a symmetrical square matrix which includes terms composed only from the cross-sectional properties and constants. The non-linear influence of the other nodal displacements are excluded from the geometric nonlinear stiffness matrices. Therefore is possible to separate the influences of different nodal displacements on the observed nodal force. This approach leads to the expression for each force which relies on the 6x6 matrices for the 3D bar element or 12 x 12 matrices for the 3D beam element. The terms in the relations can have an effect on the stability solution of the 3D space structure systems.

4.1 The 3D bar element

The general equation for the forces at the end nodes of a 3D bar element which is as follows

$$S_i = \mathbf{k_{Ei}}\ \mathbf{U} + (\mathbf{U^T}\ \mathbf{h_i}\ \mathbf{U}) + u_j\ \ (\mathbf{U^T}\ \mathbf{q}\ \mathbf{U}) + u_j\ (\mathbf{U^T}\ \mathbf{g}\ \mathbf{U}) \tag{14}$$

where index j relies on the index i of the evaluated force as j=i+3 for i=2,3 and
j=i-3 for i= 5,6

u_i, u_j .... are the nodal displacements (scalar quantities) which have the major effect on the corresponding force S_i

k_{Ei} ....is the i-th row of the elastic stiffness matrix,

$\mathbf{h_i}$ ....are square 6x6 matrices, which express the loading change during the load step (corresponds to the well known geometrical stiffness matrix $\mathbf{k_G}$)

U.............. is the 6x1 vector of node displacements

$\mathbf{U^T}$............. is simply the transpose of vector U

q, g ....are square 6x6 matrices which express the higher order terms in the longitudinal strain energy of a bar

The above equation (14) can also be written in the form:

$$\mathbf{S = S_E + S_G + S_Q} \tag{15}$$

where $\mathbf{S_E}$ are forces corresponding to the elastic linear calculation, $\mathbf{S_G}$ are forces which correspond to the geometrically non-linear behavior of the structure (previously expressed by the geometrically stiffness matrix), and last part $\mathbf{S_Q}$, expresses the influence of the higher order terms. This form clearly demonstrates the influence of the nodal deflections which are acting perpendicularly to the member axis. Matrices for the 3D bar element can be obtain from the 3D beam element matrices of next section by assuming curvature (7) is zero. These conditions are described in the following equations

$$\frac{\partial u_y}{\partial x} = \frac{1}{L}(u_8 - u_5), \quad \frac{\partial^2 u_y}{\partial x^2} = 0, \quad \frac{\partial u_z}{\partial x} = \frac{1}{L}(u_8 - u_5), \quad \frac{\partial^2 u_{yz}}{\partial x^2} = 0 \tag{16}$$

4.2 The 3D beam element

More complicated expression result for a 3D beam element. The derivation procedure is similar to that of a 3D bar element. The general equation for nodal forces applied to a 3D beam, which represent shear, axial force and bending, is

$$S_i = k_{Ei}\ \mathbf{U} + (\mathbf{U}^T\ {}^7\mathbf{h_i}\ \mathbf{U}) + (\mathbf{U}^T\ {}^1\mathbf{h_i}\ \mathbf{U}) \tag{17}$$
$$+\ u_i\ (\mathbf{U}^T\ {}^i\mathbf{e1}\ \mathbf{U}) + u_{i+6}\ (\mathbf{U}^T\ {}^{i+6}\mathbf{e2}\ \mathbf{U}) + u_j\ (\mathbf{U}^T\ {}^j\mathbf{e3}\ \mathbf{U}) + u_{j+6}\ (\mathbf{U}^T\ {}^{j+6}\mathbf{e4}\ \mathbf{U})$$

where ${}^7\mathbf{h_i} = -{}^1\mathbf{h_i}$ are square 12x12 matrices which express the loading change during the loading step (corresponds to, the well known, geometrical stiffness matrix $\mathbf{k_G}$). Superscript 7 or 1 express the influence of 7th or 1st node deflection as a major influence

k_{Ei} ....is the corresponding row of the elastic stiffness matrix

U..........................is the 12x1 vector of nodal displacements

${}^i\mathbf{e1}, {}^{i+6}\mathbf{e2}, {}^j\mathbf{e3}, {}^{j+6}\mathbf{e4}$ .. are square 12x12 matrices which express the influence of higher order terms in the axial strain energy expression of a 3D beam element

$u_i, u_j, u_{i+6}, u_{j+6}$.......... are the node displacements (scalar quantities), which have a major influence on the corresponding force S_I, where index i is 2 and index j is 6 for the forces S_2, S_8, S_6, S_{12}, index i is 3 and index j is 5 for the forces S_3, S_9, S_5, S_{11},

Expressions for the torsional moments S_4, S_{10} are slightly different. These forces represent Saint Venant's torsion. The matrix equation for the nodal torsional moment is as follows:

$$S_i = \mathbf{k_{Ei}}\ \mathbf{U} + (\mathbf{U}^T\ {}^7\mathbf{h_i}\ \mathbf{U}) + (\mathbf{U}^T\ {}^1\mathbf{h_i}\ \mathbf{U}) + u_i\ (\mathbf{U}^T\ {}^i\mathbf{g1}\ \mathbf{U}) + u_{i+6}\ (\mathbf{U}^T\ {}^{i+6}\mathbf{g2}\ \mathbf{U}) \tag{18}$$

where index *i* is either 4 or 10.

Matrices **g1** and **g2** for the torsional moment S_4, at the near end of the element include the same terms as matrices **g1** and **g2** for the torsional moment S_{10}, at the far end of the beam element, except all these terms are negative. The basic difference between the expressions for bending moments and shear forces is in the non-linear influence of the governing deflection u_i which is separated out of the matrix equation as a factor. The torsional moments are influenced only by the torsional deflection u_4 and u_{10}. Equation (17) or (18) can be expressed in a form similar to that of equation (15)

$$\mathbf{S} = \mathbf{S_E} + \mathbf{S_G} + \mathbf{S_Q} \tag{19}$$

The 3D beam expression for forces which correspond to the geometrically non-linear terms, are more complex then these for the 3D bar element. Forces $\mathbf{S_G}$ which corresponds to the forces which were calculated with the geometrically stiffness matrix $\mathbf{k_G}$ are now divided to two parts $(\mathbf{U}^T\ {}^7\mathbf{h_i}\ \mathbf{U})$ and $(\mathbf{U}^T\ {}^1\mathbf{h_i}\ \mathbf{U})$. The first part $(\mathbf{U}^T\ {}^7\mathbf{h_i}\ \mathbf{U})$ express the influence of the deflections at the far end of the element to the solved nodal force S_i. The second part $(\mathbf{U}^T\ {}^1\mathbf{h_i}\ \mathbf{U})$ express the influence of the deflections at the near end of the element to the solved nodal force S_i. For the force S_1 the matrix ${}^1\mathbf{h_1} = \mathbf{0}$. Similarly for the force S_7 the matrix ${}^7\mathbf{h_7} = \mathbf{0}$. For the force S_1 and S_7 are matrices ${}^7\mathbf{h_1} = -{}^1\mathbf{h_7}$ and the terms reminds terms in the geometrically stiffness matrix $\mathbf{k_G}$. Thus, the effect of the geometrically non-linear behavior expressed by the approximate formula using the geometrical stiffness matrix is good for axial force, but the other non-linear influences to the shear forces and bending and torsional moments are not taken into the account. These terms are important for the stability solutions of a 3D system. The matrix ${}^7\mathbf{h_1}$ is shown in equation (20). To show all the matrices is beyond the scope of this paper. Only the main matrices are shown here. All matrices are published at[12]. The terms EA can be factored out of all the stiffness matrices, E is the Young's modulus and A is the cross-sectional area. The matrices ${}^{7}\mathbf{h}_{3}$ and ${}^{1}\mathbf{h}_{3}$ which correspond to the shear force S_3, are shown as an example of the matrices of other forces. All the other matrices, ${}^{7}\mathbf{h}_{j}$ and ${}^{1}\mathbf{h}_{j}$, have corresponding occupancy of nonzero terms in the same rows and columns as these two matrices. The values of terms are different. Very interesting influences of nodal displacements of the element are derived by expressing the matrices from equation (17) and (18).

$$
{}^{7}\mathbf{h}_{1} = EA \begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-6}{10L^2} & & & & \frac{-1}{20L} & 0 & \frac{3}{5L^2} & & & & \frac{-1}{20L} \\
0 & & \frac{-6}{10L^2} & & \frac{+1}{20L} & & 0 & & \frac{+6}{10L^2} & & \frac{+1}{20L} & \\
0 & & & \frac{-(I_y+I_z)}{2L^2} & & & 0 & & & \frac{+(I_y+I_z)}{L^2} & & \\
0 & & \frac{+1}{20L} & & \frac{-1}{15} & & 0 & & \frac{-1}{20L} & & \frac{+1}{60} & \\
0 & \frac{-1}{20L} & & & & \frac{-1}{15} & 0 & \frac{+1}{20L} & & & & \frac{+1}{60} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{5L^2} & & & & \frac{+1}{20L} & 0 & \frac{-6}{10L^2} & & & & \frac{+1}{20L} \\
0 & & \frac{+6}{10L^2} & & \frac{-1}{20L} & & 0 & & \frac{-6}{10L^2} & & \frac{-1}{20L} & \\
0 & & & \frac{+(I_y+I_z)}{L^2} & & & 0 & & & \frac{-(I_y+I_z)}{2L^2} & & \\
0 & & \frac{+1}{20L} & & \frac{+1}{60} & & 0 & & \frac{-1}{20L} & & \frac{-1}{15} & \\
0 & \frac{-1}{20L} & & & & \frac{+1}{60} & 0 & \frac{+1}{20L} & & & & \frac{-1}{15}
\end{vmatrix} \qquad (20)
$$

Matrix ${}^{7}\mathbf{h}_{3}$ Matrix ${}^{1}\mathbf{h}$ (21)

$$
EA \begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & & 0 & & & & & \\
0 & & & & & & \frac{+3}{5L^2} & & & & & \\
0 & & & & & & 0 & & & & & \\
0 & & & & & & \frac{-1}{20L} & & & & & \\
0 & & & & & & 0 & & & & & \\
0 & 0 & \frac{+3}{5L^2} & 0 & \frac{-1}{20L} & 0 & 0 & 0 & \frac{-3}{5L^2} & 0 & \frac{-1}{20L} & 0 \\
0 & & & & & & 0 & & & & & \\
0 & & & & & & \frac{-3}{5L^2} & & & & & \\
0 & & & & & & 0 & & & & & \\
0 & & & & & & \frac{-1}{20L} & & & & & \\
0 & & & & & & 0 & & & & &
\end{vmatrix}
\qquad
EA \begin{vmatrix}
0 & 0 & \frac{-3}{5L^2} & 0 & \frac{-1}{20L} & 0 & 0 & 0 & \frac{-3}{5L^2} & 0 & \frac{-1}{20L} & 0 \\
0 & & & & & & & & & & & \\
\frac{-3}{5L^2} & & & & & & & & & & & \\
0 & & & & & & & & & & & \\
\frac{-1}{20L} & & & & & & & & & & & \\
0 & & & & & & & & & & & \\
0 & & & & & & & & & & & \\
0 & & & & & & & & & & & \\
\frac{-3}{5L^2} & & & & & & & & & & & \\
0 & & & & & & & & & & & \\
\frac{-1}{20L} & & & & & & & & & & & \\
0 & & & & & & & & & & &
\end{vmatrix}
$$

Matrices ${}^{i}\mathbf{e1}$, ${}^{i}\mathbf{e2}$, ${}^{i}\mathbf{e3}$, ${}^{i}\mathbf{e4}$ for shear forces S_2 and S_8 , and similarly for S_3 and S_9 , include the same terms, however they are negative for ${}^{i}\mathbf{e1}$, ${}^{i}\mathbf{e2}$, ${}^{i}\mathbf{e3}$, ${}^{i}\mathbf{e4}$ for the forces S_8 and S_9 at the far end of the element. The same is true as for the torsional moments S_4 and S_{10}. Therefore, only matrices ${}^{i}\mathbf{e1}$, ${}^{i}\mathbf{e2}$, ${}^{i}\mathbf{e3}$, ${}^{i}\mathbf{e4}$ need to be written to express the forces S_2 and force S_3. The matrices ${}^{i}\mathbf{e1}$, ${}^{i}\mathbf{e2}$, ${}^{i}\mathbf{e3}$, ${}^{i}\mathbf{e4}$ for the bending moments vary in a position and value of nonzero elements. The following are matrix expressions showing the non linear relations between internal forces and deflections with respect to the governing deflections which are as factors out of the matrices.

S_2 , shear force in *y* direction:

Matrix expression with matrix **e1** for the force S_2 (22a)

$$u_2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{+36A}{35L^3} & & & & & 0 & & & & & \\
0 & & \frac{+36A}{35L^3} & & \frac{-9A}{70L^2} & & 0 & & \frac{-36A}{35L^3} & & \frac{-9A}{70L^2} & \\
0 & & & \frac{+3(3I_y+I_z)}{5L^3} & & & 0 & & & \frac{-3(3I_y+I_z)}{5L^3} & & \\
0 & & \frac{-9A}{70L^2} & & \frac{+3A}{70L} & & 0 & & \frac{+9A}{70L^2} & & & \\
0 & & & & & \frac{+9A}{70L} & 0 & \frac{-9A}{70L^2} & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & \frac{-9A}{70L^2} & 0 & \frac{+108A}{35L^3} & & & & \frac{-9A}{70L^2} \\
0 & & \frac{-36A}{35L^3} & & \frac{+9A}{70L^2} & & 0 & & \frac{+36A}{35L^3} & & \frac{+9A}{70L^2} & \\
0 & & & \frac{-3(3I_y+I_z)}{5L^3} & & & 0 & & & \frac{+(3I_y+I_z)}{5L^3} & & \\
0 & & \frac{-9A}{70L^2} & & & & 0 & & \frac{=9A}{70L^2} & & \frac{+3A}{70L} & \\
0 & & & & & & 0 & \frac{-9A}{70L^2} & & & & \frac{+9A}{70L}
\end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}$$

Matrix expression with matrix **e2** for the force S_2: (22b)

$$u_8 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-108A}{35L^3} & & & & \frac{-9A}{70L^2} & 0 & & & & & \frac{-9A}{70L^2} \\
0 & & \frac{-36A}{35L^3} & & \frac{+9A}{70L^2} & & 0 & & \frac{+36A}{35L^3} & & \frac{+9A}{70L^2} & \\
0 & & & \frac{-3(3I_y+I_z)}{5L^3} & & & 0 & & & \frac{-3(3I_y+I_z)}{5L^3} & & \\
0 & & \frac{+9A}{70L^2} & & \frac{-3A}{70L} & & 0 & & \frac{-9A}{70L^2} & & & \\
0 & \frac{-9A}{70L^2} & & & & \frac{-9A}{70L} & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & & 0 & \frac{-36A}{35L^3} & & & & \\
0 & & \frac{+36A}{35L^3} & & \frac{-9A}{70L^2} & & 0 & & \frac{-36A}{35L^3} & & \frac{-9A}{70L^2} & \\
0 & & & \frac{+3(3I_y+I_z)}{5L^3} & & & 0 & & & \frac{-(3I_y+I_z)}{5L^3} & & \\
0 & & \frac{+9A}{70L^2} & & & & 0 & & \frac{-9A}{70L^2} & & \frac{-3A}{70L} & \\
0 & \frac{-9A}{70L^2} & & & & & 0 & & & & & \frac{-9A}{70L}
\end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}$$

Matrix expression with matrix **e3** for the force S_2: (22c)

$$
u_6 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T c
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{+27A}{70L^2} & & & & & 0 & \frac{-9A}{70L^2} & & & & \\
0 & & \frac{+9A}{70L^2} & & \frac{+3A}{70L} & & 0 & & \frac{-9A}{70L^2} & & & \\
0 & & & \frac{+(3I_y+I_z)}{20L^2} & & & 0 & & & \frac{-(3I_y+I_z)}{20L^2} & & \\
0 & & \frac{+3A}{70L} & & \frac{-A}{280} & & 0 & & \frac{+3A}{70L} & & \frac{+A}{280} & \\
0 & & & & & \frac{-A}{280} & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-9A}{70L^2} & & & & & 0 & \frac{+27A}{70L^2} & & & & \\
0 & & \frac{-9A}{70L^2} & & \frac{+3A}{70L} & & 0 & & \frac{+9A}{70L^2} & & & \\
0 & & & \frac{-(3I_y+I_z)}{20L^2} & & & 0 & & & \frac{+(3I_y+I_z)}{20L^2} & & \\
0 & & & & \frac{+A}{280} & & 0 & & & & \frac{+A}{280} & \\
0 & & & & & & 0 & & & & & \frac{+3A}{280}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}
$$

Matrix expression with matrix **e4** for the force S_2 : (22d)

$$
u_{12} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{+27A}{70L^2} & & & & & 0 & \frac{-9A}{70L^2} & & & & \\
0 & & \frac{+9A}{70L^2} & & & & 0 & & \frac{-9A}{70L^2} & & \frac{-3A}{70L} & \\
0 & & & \frac{+(3I_y+I_z)}{20L^2} & & & 0 & & & \frac{-(3I_y+I_z)}{20L^2} & & \\
0 & & & & \frac{+A}{280} & & 0 & & & & \frac{+A}{280} & \\
0 & & & & & \frac{+A}{280} & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-9A}{70L^2} & & & & & 0 & \frac{+27A}{70L^2} & & & & \\
0 & & \frac{-9A}{70L^2} & & & & 0 & & \frac{+9A}{70L^2} & & \frac{+3A}{70L} & \\
0 & & & \frac{-(3I_y+I_z)}{20L^2} & & & 0 & & & \frac{+(3I_y+I_z)}{20L^2} & & \\
0 & & \frac{-3A}{70L} & & \frac{+A}{280} & & 0 & & \frac{3A}{70L} & & \frac{-A}{280} & \\
0 & & & & & & 0 & & & & & \frac{-A}{280}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}
$$

S_3 , shear force in the z direction:

Matrix expression with matrix **e1** for the force S_3**:** (23a)

$$
u_3\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9\\ u_{10}\\ u_{11}\\ u_{12}\end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{+36A}{35L^3} & & & & \frac{+9A}{70L^2} & 0 & \frac{-36A}{35L^3} & & & & \frac{+9A}{70L^2}\\
0 & & \frac{+36A}{35L^3} & & & & 0 & & & & & \\
0 & & & \frac{+3(I_y+3I_z)}{5L^3} & & & 0 & & & \frac{-3(I_y+3I_z)}{5L^3} & & \\
0 & & & & \frac{+9A}{70L} & & 0 & & \frac{+9A}{70L^2} & & & \\
0 & \frac{+9A}{70L^2} & & & & \frac{+3A}{70L} & 0 & \frac{-9A}{70L^2} & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-36A}{35L^3} & & & & \frac{-9A}{70L^2} & 0 & \frac{+36A}{35L^3} & & & & \frac{-9A}{70L^2}\\
0 & & & & \frac{+9A}{70L^2} & & 0 & & \frac{+108A}{35L^3} & & \frac{+9A}{70L^2} & \\
0 & & & \frac{-3(I_y+3I_z)}{5L^2} & & & 0 & & & \frac{+3(I_y+3I_z)}{5L^3} & & \\
0 & & & & & & 0 & & \frac{+9A}{70L^2} & & \frac{+9A}{70L} & \\
0 & \frac{+9A}{70L^2} & & & & & 0 & \frac{-9A}{70L^2} & & & & \frac{+3A}{70L}
\end{bmatrix}
\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9\\ u_{10}\\ u_{11}\\ u_{12}\end{bmatrix}
$$

Matrix expression with matrix **e2** for the force S_3**:** (23b)

$$
u_9\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9\\ u_{10}\\ u_{11}\\ u_{12}\end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-36A}{35L^3} & & & & \frac{-9A}{70L^2} & 0 & \frac{+36A}{35L^3} & & & & \frac{-9A}{70L^2}\\
0 & & \frac{-108A}{35L^3} & & \frac{+9A}{70L^2} & & 0 & & & & \frac{+9A}{70L^2} & \\
0 & & & \frac{-3(I_y+3I_z)}{5L^3} & & & 0 & & & \frac{+3(I_y+3I_z)}{5L^3} & & \\
0 & & \frac{+9A}{70L^2} & & \frac{-9A}{70L} & & 0 & & & & & \\
0 & \frac{-9A}{70L^2} & & & & \frac{-3A}{70L} & 0 & \frac{+9A}{70L^2} & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{+36A}{35L^3} & & & & \frac{+9A}{70L^2} & 0 & \frac{-36A}{35L^3} & & & & \frac{+9A}{70L^2}\\
0 & & & & & & 0 & & \frac{-36A}{35L^3} & & & \\
0 & & & \frac{-3(I_y+3I_z)}{5L^2} & & & 0 & & & \frac{-3(I_y+3I_z)}{5L^3} & & \\
0 & & \frac{+9A}{70L^2} & & & & 0 & & & & \frac{-9A}{70L} & \\
0 & \frac{-9A}{70L^2} & & & & & 0 & \frac{+9A}{70L^2} & & & & \frac{-3A}{70L}
\end{bmatrix}
\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9\\ u_{10}\\ u_{11}\\ u_{12}\end{bmatrix}
$$

Matrix expression with matrix **e3** for the force S_3: (23c)

$$u_5\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-9A}{70L^2} & & & & \frac{-3A}{70L} & 0 & \frac{+9A}{70L^2} & & & & \\
0 & & \frac{-27A}{70L^2} & & & & 0 & & \frac{+9A}{70L^2} & & & \\
0 & & & \frac{-(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{+(I_y+3I_z)}{20L^2} & & \\
0 & & & & \frac{+A}{280} & & 0 & & & & & \\
0 & \frac{-3A}{70L} & & & & \frac{+A}{280} & 0 & \frac{+3A}{70L} & & & & \frac{-A}{280}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{+9A}{70L^2} & & & & \frac{+3A}{70L} & 0 & \frac{-9A}{70L^2} & & & & \\
0 & & \frac{+9A}{70L^2} & & & & 0 & & \frac{-27A}{70L^2} & & & \\
0 & & & \frac{+(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{-(I_y+3I_z)}{20L^2} & & \\
0 & & & & & & 0 & & & & \frac{-3A}{280} & \\
0 & & & & & \frac{-A}{280} & 0 & & & & & \frac{-A}{280}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

Matrix expression with matrix **e4** for the force S_3: (23d)

$$u_{11}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-9A}{70L^2} & & & & & 0 & \frac{+9A}{70L^2} & & & & \frac{-3A}{70L}\\
0 & & \frac{-27A}{70L^2} & & & & 0 & & \frac{+9A}{70L^2} & & & \\
0 & & & \frac{-(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{+(I_y+3I_z)}{20L^2} & & \\
0 & & & & \frac{-3A}{280} & & 0 & & & & & \\
0 & & & & & \frac{-A}{280} & 0 & & & & & \frac{-A}{280}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{+9A}{70L^2} & & & & & 0 & \frac{-9A}{70L^2} & & & & \frac{+3A}{70L}\\
0 & & \frac{+9A}{70L^2} & & & & 0 & & \frac{-27A}{70L^2} & & & \\
0 & & & \frac{+(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{-(I_y+3I_z)}{20L^2} & & \\
0 & & & & & & 0 & & & & \frac{+A}{280} & \\
0 & \frac{-3A}{70L} & & & & \frac{-A}{280} & 0 & \frac{+3A}{70L} & & & & \frac{+A}{280}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

S_5 , bending moment about the *y* axis at the near end:

Matrix expression with matrix e1 for the force S_5: (24a)

$$u_3 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-9A}{70L^2} & & & & \frac{-3A}{70} & 0 & \frac{+9A}{70L^2} & & & & \\ 0 & & \frac{-9A}{70L^2} & & & & 0 & & & & & \\ 0 & & & \frac{-(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{+(I_y+3I_z)}{20L^2} & & \\ 0 & & & & \frac{+3A}{280} & & 0 & & \frac{-3A}{70L} & & \frac{-A}{280} & \\ 0 & \frac{-3A}{70} & & & & \frac{-A}{280} & 0 & \frac{+3A}{70L} & & & & \frac{-A}{280} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{+9A}{70L^2} & & & & \frac{+3A}{70L} & 0 & \frac{-9A}{70L^2} & & & & \\ 0 & & & & \frac{-3A}{70L} & & 0 & & \frac{-27A}{70L^2} & & & \\ 0 & & & \frac{+(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{-(I_y+3I_z)}{20L^2} & & \\ 0 & & & & \frac{-A}{280} & & 0 & & & & \frac{-3A}{280} & \\ 0 & & & & & \frac{-A}{280} & 0 & & & & & \frac{-A}{280} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}$$

Matrix expression with matrix **e2** for the force S_5**:** (24b)

$$u_9 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{+9A}{70L^2} & & & & \frac{+3A}{70L} & 0 & \frac{-9A}{70L^2} & & & & \\ 0 & & \frac{+27A}{70L^2} & & \frac{-3A}{70L} & & 0 & & & & & \\ 0 & & & \frac{+(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{-(I_y+3I_z)}{20L^2} & & \\ 0 & & \frac{-3A}{70L} & & \frac{-3A}{280} & & 0 & & & & \frac{+A}{280} & \\ 0 & \frac{+3A}{70L} & & & & \frac{-A}{280} & 0 & \frac{-3A}{70L} & & & & \frac{+A}{280} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-9A}{70L^2} & & & & \frac{-3A}{70L} & 0 & \frac{+9A}{70L^2} & & & & \\ 0 & & & & & & 0 & & \frac{+9A}{70L^2} & & & \\ 0 & & & \frac{-(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{+(I_y+3I_z)}{20L^2} & & \\ 0 & & & & \frac{+A}{280} & & 0 & & & & \frac{+3A}{280} & \\ 0 & & & & & \frac{+A}{280} & 0 & & & & & \frac{+A}{280} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}$$

Matrix expression with matrix **e3** for the force S_5**:** (24c)

$$u_5\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{+3A}{70L}&&&&\frac{-A}{280}&0&\frac{-3A}{70L}&&&&\frac{+A}{280}\\
0&&\frac{+9A}{70L}&&&&0&&\frac{-3A}{70L}&&\frac{-A}{280}&\\
0&&&\frac{+(I_y+3I_z)}{15L}&&&0&&&\frac{-(I_y+3I_z)}{15L}&&\\
0&&&&\frac{+AL}{35}&&0&&&&&\\
0&\frac{-A}{280}&&&&\frac{+AL}{35}&0&\frac{+A}{280}&&&&\frac{-AL}{280}\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{-3A}{70L}&&&&\frac{+A}{280}&0&\frac{+3A}{70L}&&&&\frac{-A}{280}\\
0&&\frac{-3A}{70L}&&&&0&&\frac{+9A}{70L}&&\frac{+A}{280}&\\
0&&&\frac{-(I_y+3I_z)}{15L}&&&0&&&\frac{+(I_y+3I_z)}{15L}&&\\
0&&\frac{-A}{280}&&&&0&&\frac{+A}{280}&&\frac{+AL}{140}&\\
0&\frac{+A}{280}&&&&\frac{-AL}{280}&0&\frac{-A}{280}&&&&\frac{+AL}{420}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

Matrix expression with matrix **e4** for the force S_5 (22d)

$$u_{11}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&&&&&\frac{+A}{280}&0&&&&&\frac{+A}{280}\\
0&&&&\frac{-A}{280}&&0&&&&&\\
0&&&\frac{-(I_y+3I_z)}{60L}&&&0&&&\frac{+(I_y+3I_z)}{60L}&&\\
0&&\frac{-A}{280}&&\frac{-3AL}{280}&&0&&\frac{+A}{280}&&&\\
0&\frac{+A}{280}&&&&\frac{-AL}{280}&0&\frac{-A}{280}&&&&\frac{+AL}{420}\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&&&&&\frac{-A}{280}&0&&&&&\frac{-A}{280}\\
0&&&&\frac{+A}{280}&&0&&&&&\\
0&&&\frac{+(I_y+3I_z)}{60L}&&&0&&&\frac{-(I_y+3I_z)}{60L}&&\\
0&&&&&&0&&&&\frac{-AL}{280}&\\
0&\frac{+A}{280}&&&&\frac{+AL}{420}&0&\frac{-A}{280}&&&&\frac{-AL}{280}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

S6, bending moment about the z axis at the near end:

Matrix expression with matrix **e1** for the force S_6: (25a)

$$
u_2\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{+9A}{70L^2} & & & & & 0 & & & & & \\
0 & & \frac{+9A}{70L^2} & & \frac{-3A}{70L} & & 0 & & \frac{-9A}{70L^2} & & & \\
0 & & & \frac{+(3I_y+I_z)}{20L^2} & & & 0 & & & \frac{-(3I_y+I_z)}{20L^2} & & \\
0 & & \frac{-3A}{70L} & & \frac{-A}{280} & & 0 & & \frac{+3A}{70L} & & \frac{+A}{280} & \\
0 & & & & & \frac{-3A}{280} & 0 & \frac{-3A}{70L} & & & & \frac{+A}{280} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & \frac{-3A}{70L} & 0 & \frac{+27A}{70L^2} & & & & \\
0 & & \frac{-9A}{70L^2} & & \frac{+3A}{70L} & & 0 & & \frac{+9A}{70L^2} & & & \\
0 & & & \frac{-(3I_y+I_z)}{20L^2} & & & 0 & & & \frac{+(3I_y+I_z)}{20L^2} & & \\
0 & & & & \frac{+A}{280} & & 0 & & & & \frac{+A}{280} & \\
0 & & & & & \frac{+A}{280} & 0 & & & & & \frac{+3A}{280}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}
$$

Matrix expression with matrix **e2** for the force S_6: (25b)

$$
u_8\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-27A}{70L^2} & & & & \frac{-3A}{70L} & 0 & & & & & \\
0 & & \frac{-9A}{70L^2} & & \frac{+3A}{70L} & & 0 & & \frac{+9A}{70L^2} & & & \\
0 & & & \frac{-(3I_y+I_z)}{20L^2} & & & 0 & & & \frac{+(3I_y+I_z)}{20L^2} & & \\
0 & & \frac{+3A}{70L} & & \frac{+A}{280} & & 0 & & \frac{-3A}{70L} & & \frac{-A}{280} & \\
0 & \frac{-3A}{70L} & & & & \frac{+3A}{280} & 0 & & & & & \frac{-A}{280} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & & 0 & \frac{-9A}{70L^2} & & & & \\
0 & & \frac{+9A}{70L^2} & & \frac{-3A}{70L} & & 0 & & \frac{-9A}{70L^2} & & & \\
0 & & & \frac{+(3I_y+I_z)}{20L^2} & & & 0 & & & \frac{-(3I_y+I_z)}{20L^2} & & \\
0 & & & & \frac{-A}{280} & & 0 & & & & \frac{-A}{280} & \\
0 & & & & & \frac{-A}{280} & 0 & & & & & \frac{-3A}{280}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}
$$

Matrix expression with matrix **e3** for the force S_6: (25c)

$$u_6\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{+9A}{70L}&&&&&0&\frac{-3A}{70L}&&&&\frac{+A}{280}\\
0&&\frac{+3A}{70L}&&\frac{+A}{280}&&0&&\frac{-3A}{70L}&&\frac{-A}{280}&\\
0&&&\frac{+(3I_y+I_z)}{15L}&&&0&&&\frac{-(3I_y+I_z)}{15L}&&\\
0&&\frac{+A}{280}&&\frac{+AL}{35}&&0&&\frac{-A}{280}&&\frac{-AL}{280}&\\
0&&&&&\frac{+AL}{35}&0&&&&&\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{-3A}{70L}&&&&&0&\frac{+9A}{70L}&&&&\frac{-A}{280}\\
0&&\frac{-3A}{70L}&&\frac{-A}{280}&&0&&\frac{+3A}{70L}&&\frac{+A}{280}&\\
0&&&\frac{-(3I_y+I_z)}{15L}&&&0&&&\frac{+(3I_y+I_z)}{20L^2}&&\\
0&&\frac{-A}{280}&&\frac{-AL}{280}&&0&&\frac{+A}{280}&&\frac{+AL}{420}&\\
0&\frac{+A}{280}&&&&&0&\frac{-A}{280}&&&&\frac{+AL}{140}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

Matrix expression with matrix **e4** for the force S_6: (25d)

$$u_{12}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&&&&&\frac{+A}{280}&0&&&&&\\
0&&&&\frac{-A}{280}&&0&&&&&\\
0&&&\frac{-(3I_y+I_z)}{60L}&&&0&&&\frac{+(3I_y+I_z)}{60L}&&\\
0&&\frac{-A}{280}&&\frac{-AL}{280}&&0&&\frac{+A}{280}&&\frac{+AL}{420}&\\
0&\frac{+A}{280}&&&&\frac{-3AL}{280}&0&\frac{-A}{280}&&&&\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&&&&&\frac{-A}{280}&0&&&&&\frac{-A}{280}\\
0&&&&\frac{+A}{280}&&0&&&&\frac{+A}{280}&\\
0&&&\frac{+(3I_y+I_z)}{60L}&&&0&&&\frac{-(3I_y+I_z)}{60L}&&\\
0&&&&\frac{+AL}{420}&&0&&\frac{+A}{280}&&\frac{-AL}{280}&\\
0&&&&&&0&\frac{-A}{280}&&&&\frac{-AL}{280}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

S_{11}, bending moment about *y* axis at far end:

Matrix expression with matrix **e1** for the force S_{11}: (26a)

$$u_3\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-9A}{70L^2} & & & & & 0 & \frac{+9A}{70L^2} & & & & \frac{-3A}{70L}\\
0 & & \frac{-9A}{70L^2} & & & & 0 & & & & & \\
0 & & & \frac{-(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{+(I_y+3I_z)}{20L^2} & & \\
0 & & & & \frac{-3A}{280} & & 0 & & & & \frac{-A}{280} & \\
0 & & & & & \frac{-A}{280} & 0 & & & & & \frac{-A}{280}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{+9A}{70L^2} & & & & & 0 & \frac{-9A}{70L^2} & & & & \frac{+3A}{70L}\\
0 & & & & & & 0 & & \frac{-27A}{70L^2} & & \frac{-3A}{70L} & \\
0 & & & \frac{+(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{-(I_y+3I_z)}{20L^2} & & \\
0 & & & & \frac{-A}{280} & & 0 & & \frac{-3A}{70L} & & \frac{+3A}{280} & \\
0 & \frac{-3A}{70L} & & & & \frac{-A}{280} & 0 & \frac{-3A}{70L} & & & & \frac{+A}{280}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

Matrix expression with matrix **e2** for the force S_{11}: (26b)

$$u_9\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{+9A}{70L^2} & & & & & 0 & \frac{-9A}{70L^2} & & & & \frac{+3A}{70L}\\
0 & & \frac{+27A}{70L^2} & & & & 0 & & & & \frac{-3A}{70L} & \\
0 & & & \frac{+(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{-(I_y+3I_z)}{20L^2} & & \\
0 & & & & \frac{+3A}{280} & & 0 & & & & \frac{-A}{280} & \\
0 & & & & & \frac{+A}{280} & 0 & & & & & \frac{+A}{280}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-9A}{70L^2} & & & & & 0 & \frac{+9A}{70L^2} & & & & \frac{-3A}{70L}\\
0 & & & & & & 0 & & \frac{+9A}{70L^2} & & & \\
0 & & & \frac{-(I_y+3I_z)}{20L^2} & & & 0 & & & \frac{+(I_y+3I_z)}{20L^2} & & \\
0 & & \frac{-3A}{70L} & & \frac{+A}{280} & & 0 & & & & \frac{-3A}{280} & \\
0 & \frac{+3A}{70L} & & & & \frac{+A}{280} & 0 & \frac{-3A}{70L} & & & & \frac{-A}{280}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

Matrix expression with matrix **e3** for the force S_{11}: (26c)

$$u_5 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & \frac{+A}{280} & 0 & & & & & \frac{+A}{280} \\
0 & & & & & & 0 & & & & \frac{-A}{280} & \\
0 & & & \frac{-(I_y+3I_z)}{60L} & & & 0 & & & \frac{+(I_y+3I_z)}{60L} & & \\
0 & & & & \frac{-AL}{280} & & 0 & & & & & \\
0 & \frac{+A}{280} & & & & \frac{-AL}{280} & 0 & \frac{+A}{280} & & & & \frac{+AL}{420} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & \frac{+A}{280} & 0 & & & & & \frac{-A}{280} \\
0 & & & & & & 0 & & & & \frac{+A}{280} & \\
0 & & & \frac{+(I_y+3I_z)}{60L} & & & 0 & & & \frac{-(I_y+3I_z)}{60L} & & \\
0 & & \frac{-A}{280} & & & & 0 & & \frac{+A}{280} & & \frac{-3AL}{280} & \\
0 & \frac{+A}{280} & & & & \frac{+AL}{420} & 0 & \frac{-A}{280} & & & & \frac{-AL}{280}
\end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}$$

Matrix expression with matrix **e4** for the force S_{11}: (26d)

$$u_{11} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}^T \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{+3A}{70L} & & & & \frac{+A}{280} & 0 & \frac{-3A}{70L} & & & & \frac{-A}{280} \\
0 & & \frac{+9A}{70L} & & \frac{-A}{280} & & 0 & & \frac{-3A}{70L} & & & \\
0 & & & \frac{-(I_y+3I_z)}{15L} & & & 0 & & & \frac{-(I_y+3I_z)}{15L} & & \\
0 & & \frac{-A}{280} & & \frac{+AL}{140} & & 0 & & \frac{+A}{280} & & & \\
0 & \frac{+A}{280} & & & & \frac{-AL}{420} & 0 & \frac{-A}{280} & & & & \frac{-AL}{280} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-3A}{70L} & & & & \frac{-A}{280} & 0 & \frac{+3A}{70L} & & & & \frac{-A}{280} \\
0 & & \frac{-3A}{70L} & & \frac{+A}{280} & & 0 & & \frac{+9A}{70L} & & & \\
0 & & & \frac{-(I_y+3I_z)}{15L} & & & 0 & & & \frac{-(I_y+3I_z)}{15L} & & \\
0 & & & & & & 0 & & & & \frac{+AL}{35} & \\
0 & \frac{-A}{280} & & & & \frac{+AL}{280} & 0 & \frac{+A}{280} & & & & \frac{+AL}{35}
\end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}$$

S_{12} , bending moment about z axis at the far end:

Matrix expression with matrix **e1** for the force S_{12} (27a)

$$u_2\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\left[\begin{array}{cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{+9A}{70L^2}&&&&&0&&&&&\\
0&&\frac{+9A}{70L^2}&&&&0&&\frac{-9A}{70L^2}&&\frac{-3A}{70L}&\\
0&&&\frac{+(3I_y+I_z)}{20L^2}&&&0&&&\frac{-(3I_y+I_z)}{20L^2}&&\\
0&&&&\frac{+A}{280}&&0&&&&\frac{+A}{280}&\\
0&&&&&\frac{+3A}{280}&0&&&&&\frac{+A}{280}\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&&&&&&0&\frac{+27A}{70L^2}&&&&\frac{-3A}{70L}\\
0&&\frac{-9A}{70L^2}&&&&0&&\frac{+9A}{70L^2}&&\frac{+3A}{70L}&\\
0&&&\frac{-(3I_y+I_z)}{20L^2}&&&0&&&\frac{+(3I_y+I_z)}{20L^2}&&\\
0&&\frac{-3A}{70L}&&\frac{+A}{280}&&0&&\frac{+3A}{70L}&&\frac{-A}{280}&\\
0&&&&&\frac{+A}{280}&0&\frac{-3A}{70L}&&&&\frac{-3A}{280}
\end{array}\right]\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

Matrix expression with matrix **e2** for the force S_{12}**:** (27b)

$$u_8\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\left[\begin{array}{cccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{-27A}{70L^2}&&&&&0&&&&&\frac{-3A}{70L}\\
0&&\frac{-9A}{70L^2}&&&&0&&\frac{+9A}{70L^2}&&\frac{+3A}{70L}&\\
0&&&\frac{-(3I_y+I_z)}{20L^2}&&&0&&&\frac{+(3I_y+I_z)}{20L^2}&&\\
0&&&&\frac{-A}{280}&&0&&&&\frac{-A}{280}&\\
0&&&&&\frac{-3A}{280}&0&&&&&\frac{-A}{280}\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&&&&&&0&\frac{-9A}{70L^2}&&&&\\
0&&\frac{+9A}{70L^2}&&&&0&&\frac{-9A}{70L^2}&&\frac{-3A}{70L}&\\
0&&&\frac{+(3I_y+I_z)}{20L^2}&&&0&&&\frac{-(3I_y+I_z)}{20L^2}&&\\
0&&\frac{+3A}{70L}&&\frac{-A}{280}&&0&&\frac{-3A}{70L}&&\frac{+A}{280}&\\
0&\frac{-3A}{70L}&&&&\frac{-A}{280}&0&&&&&\frac{+3A}{280}
\end{array}\right]\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

Matrix expression with matrix **e3** for the force S_{12}: (27c)

$$u_6\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&&&&&&0&&&&&\frac{+A}{280}\\
0&&&&\frac{-A}{280}&&0&&&&\frac{-A}{280}&\\
0&&&\frac{-(3I_y+I_z)}{60L}&&&0&&&\frac{+(3I_y+I_z)}{60L}&&\\
0&&\frac{-A}{280}&&\frac{-AL}{280}&&0&&\frac{+A}{280}&&\frac{+A}{420}&\\
0&&&&&\frac{-AL}{280}&0&&&&&\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&&&&&&0&&&&&\frac{-A}{280}\\
0&&&&\frac{-A}{280}&&0&&&&\frac{+A}{280}&\\
0&&&\frac{+(3I_y+I_z)}{60L}&&&0&&&\frac{-(3I_y+I_z)}{60L}&&\\
0&&\frac{-A}{280}&&\frac{+AL}{420}&&0&&\frac{+A}{280}&&\frac{-AL}{280}&\\
0&\frac{+A}{280}&&&&&0&\frac{-A}{280}&&&&\frac{-3AL}{280}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

Matrix expression with matrix **e4** for the force S_{12}: (27d)

$$u_{12}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}^T\begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{+9A}{70L}&&&&\frac{+A}{280}&0&\frac{-3A}{70L}&&&&\\
0&&\frac{+3A}{70L}&&\frac{-A}{280}&&0&&\frac{-3A}{70L}&&\frac{+A}{280}&\\
0&&&\frac{+(3I_y+I_z)}{15L}&&&0&&&\frac{-(3I_y+I_z)}{15L}&&\\
0&&\frac{-A}{280}&&\frac{+AL}{420}&&0&&\frac{+A}{280}&&\frac{-AL}{280}&\\
0&\frac{+A}{280}&&&&\frac{+AL}{280}&0&\frac{-A}{280}&&&&\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{-3A}{70L}&&&&\frac{-A}{280}&0&\frac{+9A}{70L}&&&&\\
0&&\frac{-3A}{70L}&&\frac{+A}{280}&&0&&\frac{+3A}{70L}&&\frac{-A}{280}&\\
0&&&\frac{+(3I_y+I_z)}{60L}&&&0&&&\frac{-(3I_y+I_z)}{60L}&&\\
0&&\frac{+A}{280}&&\frac{-AL}{280}&&0&&\frac{-A}{280}&&\frac{+AL}{35}&\\
0&&&&&&0&&&&&\frac{+AL}{35}
\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\\u_{10}\\u_{11}\\u_{12}\end{bmatrix}$$

The following equations express the terms for the torsional moment at the near end of the beam element.

S_4 , torsional moment at the near end:

Matrix **g1** for the force S_4: (28a)

$$u_4\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9\\ u_{10}\\ u_{11}\\ u_{12} \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{+3(3I_y+I_z)}{5L^3} & & & & \frac{+(3I_y+I_z)}{20L^2} & 0 & \frac{-3(3I_y+I_z)}{5L^3} & & & & \frac{+(3I_y+I_z)}{20L^2}\\
0 & & \frac{+3(I_y+3I_z)}{5L^3} & & \frac{-(I_y+3I_z)}{20L^2} & & 0 & & \frac{-3(I_y+3I_z)}{5L^3} & & \frac{-(I_y+3I_z)}{20L^2} & \\
0 & & & \frac{+(K_y+K_z+2K_{zy})}{2L^3} & & & 0 & & & \frac{-(I_y+3I_z)}{6L} & & \\
0 & & \frac{-(I_y+3I_z)}{20L^2} & & \frac{+(I_y+3I_z)}{15L} & & 0 & & \frac{+(I_y+3I_z)}{20L^2} & & \frac{-3(I_y+3I_z)}{5L^3} & \\
0 & \frac{+(3I_y+I_z)}{20L^2} & & & & \frac{+(3I_y+I_z)}{15L} & 0 & \frac{-(3I_y+I_z)}{20L^2} & & & & \frac{+(3I_y+I_z)}{20L^2}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-3(3I_y+I_z)}{5L^3} & & \frac{-(3I_y+I_z)}{20L^2} & & \frac{-(3I_y+I_z)}{20L^2} & 0 & \frac{+3(3I_y+I_z)}{5L^3} & & & & \frac{-(3I_y+I_z)}{20L^2}\\
0 & & \frac{-3(I_y+3I_z)}{5L^3} & & \frac{+(I_y+3I_z)}{20L^2} & & 0 & & \frac{+3(I_y+3I_z)}{5L^3} & & \frac{+(I_y+3I_z)}{20L^2} & \\
0 & & & & & & 0 & & & \frac{+3(K_y+K_z+2K_{zy})}{2L^3} & & \\
0 & & \frac{-(I_y+3I_z)}{20L^2} & & \frac{-(I_y+3I_z)}{6L} & & 0 & & \frac{+(I_y+3I_z)}{20L^2} & & \frac{+(I_y+3I_z)}{15L} & \\
0 & \frac{+(3I_y+I_z)}{20L^2} & & & & \frac{-(3I_y+I_z)}{6L} & 0 & \frac{-(3I_y+I_z)}{20L^2} & & & & \frac{+(3I_y+I_z)}{15L}
\end{bmatrix}
\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9\\ u_{10}\\ u_{11}\\ u_{12} \end{bmatrix}$$

Matrix g2 for the force S_4: (28b)

$$u_{10}\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9\\ u_{10}\\ u_{11}\\ u_{12} \end{bmatrix}^T
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{-3(3I_y+I_z)}{5L^3} & & & & \frac{-(3I_y+I_z)}{20L^2} & 0 & \frac{+3(3I_y+I_z)}{5L^3} & & & & \frac{-(3I_y+I_z)}{20L^2}\\
0 & & \frac{-3(I_y+3I_z)}{5L^3} & & \frac{+(I_y+3I_z)}{20L^2} & & 0 & & \frac{+3(I_y+3I_z)}{5L^3} & & \frac{+(I_y+3I_z)}{20L^2} & \\
0 & & & \frac{-3(K_y+K_z+2K_{zy})}{2L^3} & & & 0 & & & \frac{-(I_y+3I_z)}{60L} & & \\
0 & & \frac{+(I_y+3I_z)}{20L^2} & & \frac{-(I_y+3I_z)}{15L} & & 0 & & \frac{-(I_y+3I_z)}{20L^2} & & \frac{+(I_y+3I_z)}{60L} & \\
0 & \frac{-(3I_y+I_z)}{20L^2} & & & & \frac{-(3I_y+I_z)}{15L} & 0 & \frac{+(3I_y+I_z)}{20L^2} & & & & \frac{+(3I_y+I_z)}{60L}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{+3(3I_y+I_z)}{5L^3} & & & & \frac{+(3I_y+I_z)}{20L^2} & 0 & \frac{-3(3I_y+I_z)}{5L^3} & & & & \frac{-(3I_y+I_z)}{20L^2}\\
0 & & \frac{+3(I_y+3I_z)}{5L^3} & & \frac{-(I_y+3I_z)}{20L^2} & & 0 & & \frac{-3(I_y+3I_z)}{5L^3} & & \frac{-(I_y+3I_z)}{20L^2} & \\
0 & & & & & & 0 & & & \frac{-(K_y+K_z+2K_{zy})}{2L^3} & & \\
0 & & \frac{+(I_y+3I_z)}{20L^2} & & \frac{+(I_y+3I_z)}{60L} & & 0 & & \frac{-(I_y+3I_z)}{20L^2} & & \frac{-(I_y+3I_z)}{15L} & \\
0 & \frac{-(3I_y+I_z)}{20L^2} & & & & \frac{+(3I_y+I_z)}{60L} & 0 & \frac{+(3I_y+I_z)}{20L^2} & & & & \frac{-(3I_y+I_z)}{15L}
\end{bmatrix}
\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5\\ u_6\\ u_7\\ u_8\\ u_9\\ u_{10}\\ u_{11}\\ u_{12} \end{bmatrix}$$

4.3. Matrix formula for the whole element

Also the matrix formulas can be expressed for the whole element, but it will be necessary to increase the size of the matrices for the 3D bar element to 36x36, and 144x144 for the 3D beam element. For the 3D bar element we can write

$$\mathbf{S = k_E\, U + v_0^T\, h\, v + (w\ v_0^T\ q\, v) - ([w_B\ w_A]\, v_0^T\, g\, v)} \qquad (29)$$

where the matrices are defined as

$\mathbf{v_0^T}$..... is 6x36 matrix composed from 6x6 matrices, where *i*-row corresponding to the force S_i includes all deflections u_1, u_2, u_3,.... u_6,, the other rows are zero

v.……...is 36x1 vector composed from vectors **U**

w...…..is diagonal square 6x6 matrix with deflections u_1, u_2, u_3.,... u_6 on the diagonal

$\mathbf{[w_B\ w_A]}$....... is matrix formed similarly as the matrix **w**, only the submatrices corresponding to the near and far end of the element are in opposite order

Similarly the equation for the 3D beam element:

$$\mathbf{S = k_E\, U + (v_0^T\ h7\ v) + (v_0^T\ h1\ v) + w_1 (v_0^T\ e1\, v) + w_2 (v_0^T\ e2\, v) + w_3 (v_0^T\ e3\, v) + w_4 (v_0^T\ e4\, v)} \qquad (30)$$

where matrices are as follows:

$\mathbf{v_0^T}$....... is 12x144 matrix composed from the12x12 matrices, where *i*-row corresponding to the force S_i includes all deflections u_1, u_2, u_3,.... u_{12},, the other rows are zero

v........…is 144x1 vector, composed from 12x1 vectors **U**

$\mathbf{w_1, w_2, w_3, w_4}$ are square diagonal 144x144 matrices. The main diagonal includes deflections which have significant effect to the corresponding force. Basically we can say, that matrices

$\mathbf{w_1, w_2}$ .... express the influence of shear deflections and torsional deflection

$\mathbf{w_3}$ and $\mathbf{w_4}$ ....express the influence of deflections due to the bending moments

h7......…is square 144 x 144 matrix with matrices $^7\mathbf{h_i}$ on the main diagonal

h1....… .is square 144 x 144 matrix with matrices $^1\mathbf{h_i}$ on the main diagonal

e1......… is square 144 x 144 matrix with matrices $^I\mathbf{e1}$ on the main diagonal

e2.....…..is square 144 x 144 matrix with matrices $^I\mathbf{e2}$ on the main diagonal

e3.......... is square 144 x 144 matrix with matrices $^I\mathbf{e3}$ on the main diagonal

e4.......... is square 144 x 144 matrix with matrices $^I\mathbf{e4}$ on the main diagonal

Let's express matrices $\mathbf{w_1, w_2, w_3, w_4}$, which include deflections with significant influence to the corresponding forces. Matrices are diagonal and position of nodal deflection on diagonal express influence of non-linear behavior to the corresponding forces.

$$\text{Matrix } \mathbf{w_1} = \text{diag}\,(0, u_2, u_3, u_4, u_3, u_2, 0, u_2, u_3, u_{10}, u_3, u_2) \qquad (31)$$
$$\text{Matrix } \mathbf{w_2} = \text{diag}\,(0, u_8, u_9, u_4, u_9, u_8, 0, u_8, u_9, u_{10}, u_9, u_8)$$
$$\text{Matrix } \mathbf{w_3} = \text{diag}\,(0, u_6, u_5, 0, u_5, u_6, 0, u_6, u_5, 0, u_5, u_6)$$
$$\text{Matrix } \mathbf{w_4} = \text{diag}\,(0, u_{12}, u_{11}, 0, u_{11}, u_{12}, 0, u_{12}, u_{11}, 0, u_{11}, u_{12})$$

It is possible to see how the far and near end deflections affect the opposite side of the element if we consider the nonlinear behavior, which in the real world actually exists. Matrices which express influence of the bending are $\mathbf{w_3}$ and $\mathbf{w_4}$. Matrices which express influence of the shear and the torsion, are $\mathbf{w_1}$, $\mathbf{w_2}$. We can easily separate influencies of each standalone deflection to the element.

5. THE INFLUENCE OF THE SHEAR ENERGY

The influence on the geometrically nonlinear behavior due to the shear energy should be also included in the expression for the energy of the element. In the space framework it is the distribution of bending and the torsional moments should be taken into the account also. For space structures are usually used tubular sections so the Saint Venant torsion express properly the behavior of the member. We can make some basic assumptions about the shear deflection and about the cross section. The energy of the element due to the shear deflection can be expressed in equation (32)

$$U = \tfrac{1}{2}\int_V \tau\gamma\, dV = \tfrac{1}{2}\int_V G\gamma^2_{xy}\, dV + \tfrac{1}{2}\int_V G\gamma^2_{xz}\, dV + \tfrac{1}{2}\int_V G\gamma^2_{yz}\, dV \tag{32}$$

We can assume that the cross section of the beam element is not deformable in their plane, therefore the longitudinal change of the deformation u_x is constant with respect to the axis z and y. With respect to these assumptions the cross sections of the beam are not deformed, they are only rotated against each other under the Saint Venant torsion. For the position at axis x, e.g. $z_0 = 0$, $y_0 = 0$ we can write

$$\frac{\partial u_x}{\partial y} = 0,\ \frac{\partial u_x}{\partial z} = 0 \qquad \gamma_{zy} = 0 \qquad \frac{\partial u_x}{\partial y_0} = \frac{\partial u_x}{\partial z_0} = 0 \tag{33}$$

We can now write the expression for the rest of the shear stress tensor as follows

$$\gamma_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial x}\cdot\frac{\partial u_z}{\partial y} \qquad \gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_y}{\partial x}\cdot\frac{\partial u_y}{\partial z} \tag{34}$$

Energy of the deformed beam due to the shear deflections can be expressed at equation (35)

$$U = \frac{G}{2}\int_L\int_A \left[\left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\cdot\frac{\partial u_z}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial x}\cdot\frac{\partial u_y}{\partial z}\right)^2 + 2\left(\frac{\partial u_y}{\partial x}\right)\left(\frac{\partial u_z}{\partial x}\cdot\frac{\partial u_z}{\partial y}\right) + 2\left(\frac{\partial u_z}{\partial x}\right)\left(\frac{\partial u_y}{\partial x}\cdot\frac{\partial u_y}{\partial z}\right)\right] dx\, dA \tag{35}$$

After the same procedure which was shown and explain before, we can receive matrix expression for the nodal forces with respect to the energy spent for the shear deformation due to the Saint Venant torsion. The terms in the matrices are similar to the values at the stiffness matrix with respect to the axial deformation. For the materials with relatively large shear modulus is therefore necessary to include the effect to the analysis. Final matrix equations for the nodal forces is similarly as previously, different for the forces due to shear and bending and different for the forces due to the torsion S_4 , and S_{10}. We have for the first group of the forces the equation (36)

$$\mathbf{S} = \frac{GA}{2}\ \{\ \mathbf{e}\ \mathbf{U}\ +\ \tfrac{1}{3}\,\mathbf{e}\ \mathbf{U}\ \mathbf{U}^{\mathbf{T}}\ \mathbf{c}\ \mathbf{U}\ \} \tag{36}$$

and for the torsional forces we have

$$S_4 = \frac{GA}{2}\ {}^{4}\mathbf{d}\ \mathbf{U}\ \mathbf{U}^{\mathbf{T}}\ \mathbf{r}\ \mathbf{U}, \qquad S_{10} = \frac{GA}{2}\ {}^{10}\mathbf{d}\ \mathbf{U}\ \mathbf{U}^{\mathbf{T}}\ \mathbf{r}\ \mathbf{U}, \tag{37}$$

where matrix ${}^{4}\mathbf{d}$ and matrix ${}^{4}\mathbf{d}$ are diagonal square 12x12 matrices and on the main diagonal there are only two values. This is value 2 and 1 which correspond to the deflection u_4 and u_{10} for the matrix ${}^{4}\mathbf{d}$ and opposite for the matrix ${}^{10}\mathbf{d}$. Matrices **e**, **r** are so called „shear torsional stiffness„ 12x12 matrices, and **c** is the matrix which express the influence of the torsional moments at one end to the torsional moment at the other end. The values of the matrix terms are similar to the values of matrices which are expressing the influence of the axial forces. The only difference is in the multiplication by shear modulus. For the steel it is roughly 1/3 of the Young's modulus. This is still significant effect interesting for the behavior in space. The stiffness matrices **e** , **r** and the matrix **c** are as follows:

Matrix **e** Matrix **r** (38)

$$\begin{vmatrix}
0 &&&&&&&&&&& \\
& +\frac{12}{5L} &&&& +\frac{1}{5} && +\frac{12}{5L} &&&& +\frac{1}{5} \\
&& +\frac{12}{5L} && -\frac{1}{5} &&&& +\frac{12}{5L} && -\frac{1}{5} & \\
&&& \frac{2(I_y+I_z)}{AL} &&&&&& -\frac{2(I_y+I_z)}{AL} && \\
&& -\frac{1}{5} && +\frac{4}{15}L &&&& +\frac{1}{5} && -\frac{L}{15} & \\
& +\frac{1}{5} &&&& +\frac{4}{15}L && -\frac{1}{5} &&&& -\frac{L}{15} \\
&&&&&&&&&&& \\
& +\frac{12}{5L} &&&& -\frac{1}{5} && +\frac{12}{5L} &&&& -\frac{1}{5} \\
&& +\frac{12}{5L} && +\frac{1}{5} &&&& +\frac{12}{5L} && +\frac{1}{5} & \\
&&& -\frac{2(I_y+I_z)}{AL} &&&&&& \frac{2(I_y+I_z)}{AL} && \\
&& -\frac{1}{5} && -\frac{L}{15} &&&& +\frac{1}{5} && +\frac{4}{15}L & \\
& +\frac{1}{5} &&&& -\frac{L}{15} && -\frac{1}{5} &&&& +\frac{4}{15}L
\end{vmatrix}
\qquad
\begin{vmatrix}
0 &&&&&&&&&&& \\
& +\frac{2}{5} &&&& +\frac{L}{30} && +\frac{2}{5} &&&& -\frac{L}{30} \\
&& +\frac{2}{5} && -\frac{L}{30} &&&& +\frac{2}{5} && -\frac{L}{30} & \\
&&& 0 &&&&&& 0 && \\
&& -\frac{L}{30} && +\frac{2}{45}L^2 &&&& +\frac{L}{30} && -\frac{L^2}{90} & \\
& +\frac{L}{30} &&&& +\frac{2}{45}L^2 && -\frac{L}{30} &&&& -\frac{L^2}{90} \\
&&&&&& 0 &&&&& \\
& +\frac{2}{5} &&&& -\frac{L}{30} && +\frac{2}{5} &&&& -\frac{L}{30} \\
&& +\frac{2}{5} && +\frac{L}{30} &&&& +\frac{2}{5} && +\frac{L}{30} & \\
&&& 0 &&&&&& 0 && \\
&& -\frac{L}{30} && -\frac{L^2}{90} &&&& +\frac{L}{30} && +\frac{2}{45}L^2 & \\
& -\frac{L}{30} &&&& -\frac{L^2}{90} && -\frac{L}{30} &&&& +\frac{2}{45}L^2
\end{vmatrix}$$

Matrix **c** Matrix ${}^{4}\mathbf{d}$ (39)

$$\begin{vmatrix}
0 &&&&&&&&&&& \\
& 0 &&&&&&&&&& \\
&& 0 &&&&&&&&& \\
&&& 1 &&&&&& +\frac{1}{2} && \\
&&&& 0 &&&&&&& \\
&&&&& 0 &&&&&& \\
&&&&&& 0 &&&&& \\
&&&&&&& 0 &&&& \\
&&&&&&&& 0 &&& \\
&&& +\frac{1}{2} &&&&&& 1 && \\
&&&&&&&&&& 0 & \\
&&&&&&&&&&& 0
\end{vmatrix}
\qquad
\begin{vmatrix}
0 &&&&&&&&&&& \\
& 0 &&&&&&&&&& \\
&& 0 &&&&&&&&& \\
&&& 2 &&&&&&&& \\
&&&& 0 &&&&&&& \\
&&&&& 0 &&&&&& \\
&&&&&& 0 &&&&& \\
&&&&&&& 0 &&&& \\
&&&&&&&& 0 &&& \\
&&&&&&&&& 1 && \\
&&&&&&&&&& 0 & \\
&&&&&&&&&&& 0
\end{vmatrix}$$

To transform matrix expressions to the global coordinate system we can introduce basic transformation matrix as is

$$\mathbf{t^1} = \begin{vmatrix} \cos(\alpha_{xX}) & \cos(\alpha_{yX}) & \cos(\alpha_{zX}) \\ \cos(\alpha_{xY}) & \cos(\alpha_{yY}) & \cos(\alpha_{zY}) \\ \cos(\alpha_{xZ}) & \cos(\alpha_{yZ}) & \cos(\alpha_{zZ}) \end{vmatrix} \tag{40}$$

which transform forces at one end of the element. The transformation matrix for the whole element is square matrix with matrices $\mathbf{t^1}$ on main diagonal. We can transform all vectors and matrices which includes deflections according to the relation

$$\mathbf{D_G} = \mathbf{T\,D_L} \tag{41}$$

where $\mathbf{D_G}$ is the matrix of phenomena in the global coordinate system, $\mathbf{T}$ is the transformation matrix and $\mathbf{D_L}$ is matrix of the phenomena in the local coordinate system.

6. CONCLUSION

The derivation of the complete equation has been presented for the geometric non-linear behavior of the 3D beam with six degrees of the freedom at each end of the element. All matrices which express the material and cross-sectional influence on the end deflections of the element, are square and symmetrical. This is in agreement with the Betti's law. The influence of the other nodal motions, which express the non-linear behavior, are outside the matrices. However the operation has an effect on the size of matrices for the whole element. Each nodal force is dependent on all nodal motions in a rather complex way. Therefore, the effect is expressed by the size of the 12x12 matrices for each nodal force. Matrix equations for the whole element must deal with 144x144 matrices. The iterative procedure with the possibility to solve separately effects of each force is recommended. The numerical example of the procedure is given at[10]. However, for practical programming it is more useful to use non-matrix expressions. Preparing these expressions in the global coordinate system is relatively easy with the help of appropriate mathematical software.

ACKNOWLEDGEMENT

The author would like to express his thanks to the Czech Grant office namely to grant 103/97/K003 for the financial support for this work.

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